Abstract:
This paper presents a theoretical framework for the optimization of a class of control dependent diffusions, namely, continuous-time stochastic systems in which the absolute value of the control variations increase the state uncertainty (CVIU systems). It is shown that this class of systems can be used in the modelling of complex stochastic systems for which the dynamics are not completely known. Controlled Itô diffusion processes describes the state path of the systems, and it is necessary to solve the Hamilton-Jacobi-Bellman equation of a nonlinear system in order to achieve the optimal solution. In addition, tools from nonsmooth analysis indicated the existence of a region in the state space where the optimal control action is characterized by no variation, yielding periods of inaction in the control policy. This behavior is somehow expected from the cautionary nature of controlling underdetermined systems, but here it is obtained directly from optimality.

Keywords: Nonlinear Control Systems, Dynamic Programming, Optimization Problems, Stochastic Modelling, Uncertain Dynamic Systems.

1. INTRODUCTION

In many real world situations it can be difficult to obtain accurate models of the stochastic systems by traditional techniques of the system identification theory. This difficulty can occur for several reasons, particularly in critical situations where it is not possible to experiment with the system (like in the case of controlling the national economy or a medical intervention in the human body). In this scenario, system modeling has to rely on few scattered historical data which typically offers limited information, so the models obtained present a considerable uncertainty in its parameters. Consequently, it is important to develop controllers capable of acting effectively on a poorly known dynamics.

In order to refer to optimal control, it is necessary to perfectly know the dynamics of the system and the noise distribution involved, which is far from the scenario just outlined. When the dynamics is not perfectly known there is a trade off, and one can no longer speak of optimal solution but has to give way to worst-case analysis, yielding the sense of robust control.

If an underdetermined system is operating around an equilibrium point, its dynamics can be studied from variations of the state and the control with respect to the nominal values of the equilibrium. A common approach in this case is to obtain a linear model to describe it, and its accuracy will be restricted to a narrow range of values of the state and the control. Moreover, due to nonlinearities or other undetermined dynamics of the system, as a consequence of the control action, the system could be driven to regions where the linear approximation error becomes very large. Hence, a connection can be made between deviations in the model and the uncertainty generated by variations in the magnitude of the control signal.

Inspired by this connection, an alternative approach was presented in (Souto and do Val, 2012) for the...
continuous-time case. It is based on a linear model with an extra noise term modulated by variations of the control signal. The addition of this term generates a new class of stochastic system in which the control variation increases the uncertainty (CVIU systems, for short). In the attempt to solve the optimization problem involving CVIU systems one identifies a region of inaction, i.e., a region in the state space where the optimal solution is precautionary, in the sense that inside it, one should not change the control action applied so far. It yields periods of inaction in the control policy and brings up the idea of a cautionary control.

Cautionary controls are studied in the area of economic models dedicated to understand situations in which decisions must be taken in scenarios where the dynamics are not fully known (Stokey, 2009). In addition, we can identify other examples for which this characteristic is present (see discussion in Section 2).

The mathematical issues involved in optimizing these CVIU systems were first studied in the discrete-time (Calmon et al., 2009). The aim of this paper is then to extend the theory of CVIU systems and provide a mathematical framework for this class of systems in continuous-time using diffusion models.

This paper is divided as follows: Section 2 presents the definition of a continuous-time (single input) CVIU system, states the control problem and gives two possible scenarios of application. Section 3 introduces an essential result for this study, that is, the optimal cost function inherits convexity under some weak assumptions. Section 4 presents the traditional dynamic programming method, but it includes some tools from nonsmooth analysis in order to deal with the non differentiability of the Hamiltonian. Finally, this paper is finalized with a summary of the results in Section 5.

2. CVIU MODELING

2.1 The Single Input CVIU System

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) be a filtered probability space satisfying the usual conditions, on which a \(m\)-dimensional Brownian motion \(W = \{W_t\}_{t \geq 0}\) is defined. We assume that \(\mathcal{F}_t = \sigma\{W_r : 0 \leq r \leq t\}\) is the \(P\)-augmentation of the natural filtration of \(W\). Consider a system governed by the following time-homogeneous Itô's stochastic differential equation (SDE)

\[
dZ_t = G(Z_t, u_t) dt + \omega_t dW_t, \quad t \geq 0,
\]

where \(t \rightarrow Z_t \in \mathbb{R}^n\) is the state path, \(Z_0 = z\) is the initial state and \(t \rightarrow \omega_t \in \mathbb{R}^{n \times m}\) is the diffusion coefficient. Also, \(t \rightarrow u_t \in \mathbb{U}\) is the control, where \(\mathbb{U} \subset \mathbb{R}\) is a given Borel set. Let \(T\) be a strictly positive real number.

An admissible strict control is a \(\mathcal{F}_t\)-adapted process \(t \rightarrow u_t\) with values in \(\mathbb{U}\) such that

\[
E \left[ \sup_{t \in [0,T]} |u_t|^2 \right] < \infty.
\]

Under this set of admissible strict controls, for any \(z \in \mathbb{R}^n\), equation (1) admits a unique solution \(t \rightarrow Z_t\) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) in the sense of probability law.

Suppose that \(G : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n\) is uncertain and the system is operating near a certain equilibrium point \((x_e, u_e)\), i.e., its original state equation (1) can be rewritten as

\[
dZ_t = G(x_t + X_t, u_t + v_t) dt + \omega_t dW_t, \quad t \geq 0,
\]

where \(t \rightarrow X_t\) and \(t \rightarrow v_t\) describe the variations of the state and the control, respectively.

Noting that \(dZ_t = dX_t\), the uncertain system (1) can be approximated by variations \(X_t\) around the equilibrium

\[
X_t = Z_t - x_e,
\]

and a possible way to model \(X_t\) under a partial knowledge of the system dynamics is to use a linear model for it, as follows

\[
dx_t = (A_t X_t + B_t v_t) dt + \omega_t dW_t, \quad t \geq 0,
\]

where \(t \rightarrow A_t\) and \(t \rightarrow B_t\) have compatible dimensions. Nevertheless, the alternative model proposed here demands the addition of an extra noise term modulated by the absolute value of the control variations \(v_t\), which allows us to write

\[
\omega_t = (\sigma_1_t + \sigma_2_t |v_t|) dW_t,
\]

Hence, if we describe the variation process \(t \rightarrow X_t\) in (2) by means of the SDE (4)–(5), we have a CVIU model of the process \(t \rightarrow Z_t\). The matrix functions \(t \rightarrow A_t\) and \(t \rightarrow B_t\) represent the known local model of the original process. The additional term \(\sigma_2_t |v_t| dW_t\) represents the error generated in the state, induced by deviations of higher orders due to variations \(v_t\) around the nominal value of the control. Thus, the CVIU model expresses, with this extra term, the uncertainties that are not taken into account by the linear model.

2.2 The Control Problem

Suppose that the system (4)–(5) has all matrix functions \(t \rightarrow A_t \in \mathbb{R}^{n \times n}, t \rightarrow B_t \in \mathbb{R}^{n}, t \rightarrow \sigma_1_t \in \mathbb{R}^{n \times m}\) and \(t \rightarrow \sigma_2_t \in \mathbb{R}^{n \times m}\) \(t\)-continuous and deterministic an let \(X_0 = x\) be its initial state. Also, consider that the system performance is evaluated by the cost functional \(J\) given by the expected value of a criterion of Bolza type

\[
J(x,v(\cdot)) = E^x \left[ \int_0^T f(t, X_t, v_t) dt + g(X_T) \right],
\]

where \(f : [0,T] \times \mathbb{R}^n \times \mathbb{U} \rightarrow \mathbb{R}\) and \(g : \mathbb{R}^n \rightarrow \mathbb{R}\) are given continuous functions, and \(E^x[\cdot]\) denotes the expectation from \(t = 0\) and \(X_0 = x\). Furthermore, \(g(x)\) is a convex function, and \(f(t, x, v)\) is continuously differentiable and locally Lipschitz in \(v\), and also convex in both variables \(x\) and \(v\).
We seek a control signal that is a Markov function \(v_t = v(t, X_t(\omega))\) and thus it is a \(\mathcal{F}_t\)-valued \(\mathcal{F}_t\)-adapted measurable process. So, the control problem consists of finding \(v^*\) such that it is optimal in the sense that
\[
v^* = v^*(\cdot, \omega) = \arg \min_{v(\cdot)} J(x, v(\cdot))
\]
(7)
Thus, if such a control exists, it is called optimal control and it has the property that
\[
J^*(x) = \inf_{v(\cdot)} J(x, v(\cdot)) = J(x, v^*),
\]
and \(J^*(x)\) is called the optimal cost.

2.3 Scenarios of Application

Optimal Harvesting in an Unknown Environment
Suppose that a fish population is harvested under a controlled rate \(u_t\) and the amount of biomass \(Z_t\) (mass of the biological species) of the harvested species at time \(t\) can be estimated with reasonable precision by the stochastic differential equation
\[
dZ_t = (r_0(1 - Z_t/K)Z_t - u_t) dt + \sigma dW_t, \quad t \geq 0,
\]
(9)
where \(X(0) = x > 0\) is the initial biomass, \(r_0 > 0\) is the constant intrinsic growth rate in the absence of intra-specific competition, \(K > 0\) is the constant biomass carrying capacity that reflects the size of the population that the environment can support in absence of harvesting and other factors, \(\sigma\) is the diffusion coefficient and \(W_t\) is the standard Brownian motion. The natural growth function \(f(x) = r_0(1 - x/K)x\) is the logistic function. A realistic assumption is that the specific values of the parameters model \(r_0\) and \(K\) are not known, and a CVIU model (4) can be used to represent (9) where \(A\) and \(B\) are obtained from linearization near an equilibrium point, and \(\sigma = \omega + \sigma_2|v_t|\), for some \(\sigma_2 > 0\).

Optimal Investment of a Firm
Consider a firm that uses capital and an adjustable input to produce a nonstorable output (Abel and Eberly, 1994). Capital is acquired by undertaking gross investment at rate \(u_t\), and the capital stock \(Z_t\) depreciates at a fixed proportional rate \(\delta\), in a way that
\[
dZ_t = (u_t - \delta Z_t) dt, \quad t \geq 0.
\]
(10)
At each instant \(t\), the firm chooses the input to maximize the value of its revenue minus expenditures on this input. Let \(\pi(Z_t, \varepsilon_t)\) denote the instantaneous operating profit at time \(t\), where \(\varepsilon_t\) is a random variable that could represent randomness in the price of the adjustable input or in the demand facing the firm. Assume that \(\varepsilon_t\) evolves according to a diffusion process
\[
d\varepsilon_t = \mu dt + \sigma dW_t, \quad t \geq 0,
\]
(11)
where \(W_t\) is the standard Brownian motion. When the firm undertakes gross investment, it incurs on costs that we can describe by a function \(c(u_t, Z_t)\) that is strictly convex in \(u_t\) and differentiable everywhere except possibly at \(u_t = 0\), where it takes the value 0. The firm chooses investment to maximize the expected value of the difference between the instantaneous operating profit \(\pi(Z_t, \varepsilon_t)\) and the investment cost \(c(u_t, Z_t)\). In this model, the non-differentiability with respect to the control \(u_t\) arises in connection with the net profit, and can be modeled as CVIU.

3. CONVEXITY CHARACTERIZATION

We aim at the dynamic programming (DP) method, and in a preliminary step we are interested in characterizing the optimal cost-to-go function, defined as
\[
J^*(t, x) = \inf_{v(\cdot)} E^{t,x} \left[ \int_t^T f(r, X_r, v_r) dr + g(X_T) \right],
\]
(12)
where \(E^{t,x}[\cdot]\) denotes the expectation from time \(t\) and \(X_t = x\). The following propositions are important for the convexity characterization of \(J^*(t, x)\).

Proposition 1. (Davis, 1977; Kallianpur, 1980) Let \(t \to X_t \in \mathbb{R}^n\) be an Itô diffusion such as (4) where \(v_t \in \mathcal{U}\) is \(\mathcal{F}_t\)-adapted. Thus, if \(X_0 = x\) is a Gaussian random variable, independent of \(W_t\) in the interval \(0 \leq t \leq T\), then the process \(X_t\) will be Gaussian.

Proposition 2. (Scarsini, 1998) Let \(X\) and \(Y\) be two multidimensional random variable where \(X \sim N(\mathbb{X}, \Sigma_X)\) and \(Y \sim N(\mathbb{Y}, \Sigma_Y)\), i.e., \(X\) and \(Y\) has Gaussian distributions. Thus, \(\mathbb{X} = \mathbb{Y}\) and \(\Sigma_Y - \Sigma_X \subsetneq 0 \implies E[\varphi(X)] \leq E[\varphi(Y)]\) for all convex function \(\varphi\).

Proposition 3. (Boyd and Vandenberghe, 2004, p 79) If \(f(x, y)\) is a convex function in \(x\) for each \(y \in \mathbb{S}\), then the function \(\varphi\) defined as
\[
\varphi(x) = \int_{\mathbb{S}} f(x, y) dy
\]
is convex in \(x\) (provided the integral exists).

Using the above, we can now prove a very intuitive result: if we have two diffusion processes with the same mean and evaluated by the same convex cost functional, then the process that has the greatest dispersion also has the highest cost (in terms of expected value). This is stated in the following lemma.

Lemma 4. Let \(t \to X_t \in \mathbb{R}^n\) and \(t \to \tilde{X}_t \in \mathbb{R}^n\) be two controlled Itô diffusion processes, such as
\[
dX_t = (A_tX_t + B_t v_t) dt + \sigma_t dW_t, \quad X_0 = x_0, \quad \tilde{X}_0 = \tilde{x}_0,
\]
\[
d\tilde{X}_t = (A_t\tilde{X}_t + B_t v_t) dt + \tilde{\sigma}_t dW_t, \quad \tilde{X}_0 = \tilde{x}_0,
\]
where \(v_t \in \mathcal{U}\) is \(\mathcal{F}_t\)-adapted. If \(f(t, x, v)\) is a convex function in \(x\), the initial state \(x_0\) is a Gaussian random
variable independent of $W_t$ for $0 \leq t \leq T$ and the diffusion coefficient $\tilde{\sigma}_t$ is larger than the diffusion coefficient $\sigma_t$, in the sense of $\tilde{\sigma}_t^2 > \sigma_t^2$, $0 \leq t \leq T$.

\[
E^0 \left[ \int_t^T f(t, X_t, v_t) \, dt \right] \leq E^0 \left[ \int_t^T f(t, \bar{X}_t, v_t) \, dt \right].
\]  

(15)

**PROOF.** According to Proposition 1, the processes $t \to X_t$ and $t \to \bar{X}_t$ are Gaussian, which implies that the random variables $X_t$ and $\bar{X}_t$ have normal distribution with mean $E[X_t]$ and $E[\bar{X}_t]$ and covariance matrices $\sigma_t^2$ and $\tilde{\sigma}_t^2$, respectively, for each instant $t$. Hence, the Ito’s integral representation of the processes (13) and (14) can be written as

\[
X_t = x_0 + \int_0^t (A_t X_r + B_t v_r) \, dr + \int_0^t \sigma_r \, dW_r, \quad (16)
\]

\[
\bar{X}_t = x_0 + \int_0^t (A_t \bar{X}_r + B_t v_r) \, dr + \int_0^t \tilde{\sigma}_r \, dW_r.
\]  

(17)

and, using Itô calculus, we have

\[
E[X_t] = E[x_0] + E \left[ \int_0^t (A_t X_r + B_t v_r) \, dr \right],
\]  

(18)

\[
E[\bar{X}_t] = E[x_0] + E \left[ \int_0^t (A_t \bar{X}_r + B_t v_r) \, dr \right].
\]  

(19)

So, both processes $X_t$ and $\bar{X}_t$ have the same mean and, consequently, the combination of the Propositions 2 and 3 leads to the result (15), completing the proof.

In view of Lemma 4, we can state the main result of this section, i.e., the optimal cost-to-go function (12) will be convex if we have the functions $f$ and $g$ convex too. The following theorem states this.

**Theorem 5.** Let $t \to X_t$ be the controlled Itô diffusion process (4)–(5) evaluated by (6). Thus, if $f(t, x, v)$ is convex in both variables $x$ and $v$, and if $g(x)$ is convex in $x$, then the optimal cost-to-go $J^*(t, x)$ will also be a convex function in $x$.\n
**PROOF.** Let $t \to X_t^{(a)} \in \mathbb{R}^n$ and $t \to X_t^{(b)} \in \mathbb{R}^n$ be any two distinct state paths of CVIU systems with initial states $X_{0}^{(a)} = x^{(a)}$ and $X_{0}^{(b)} = x^{(b)}$, respectively, and $\mathcal{F}_t$-adapted control signals $t \to v_t^{(a)} \in \mathbb{U}$ and $t \to v_t^{(b)} \in \mathbb{U}$. Thus, using the stochastic integral formulation, we have

\[
X_t^{(a)} = x^{(a)} + \int_0^t (A_t X_r^{(a)} + B_t v_r^{(a)}) \, dr + \int_0^t \left( \sigma_{1,r} + \sigma_{2,r} |v_r^{(a)}| \right) \, dW_r, \quad (20a)
\]

\[
X_t^{(b)} = x^{(b)} + \int_0^t (A_t X_r^{(b)} + B_t v_r^{(b)}) \, dr + \int_0^t \left( \sigma_{1,r} + \sigma_{2,r} |v_r^{(b)}| \right) \, dW_r, \quad (20b)
\]

for $t \geq 0$. Let us define

\[
x^* = \alpha x^{(a)} + \beta x^{(b)} \text{ and } v_t^* = \alpha v_t^{(a)} + \beta v_t^{(b)},
\]  

(21)

so that we can create an auxiliary state path $t \to X_t^t \in \mathbb{R}^n$ with initial condition and control given by (21).

\[
X_t^{c} = x^* + \int_0^t \left[ A_t \left( \alpha X_r^{(a)} + \beta X_r^{(b)} \right) + B_t v_r^* \right] \, dr + \int_0^t \left[ \sigma_{1,r} + \sigma_{2,r} |v_r^*| \right] \, dW_r.
\]  

(22)

Now, let us define $t \to \tilde{X}_t^c$ by the convex combination of the paths $t \to X_t^{(a)}$ and $t \to X_t^{(b)}$, i.e.,

\[
\tilde{X}_t^c = \alpha X_t^{(a)} + \beta X_t^{(b)}, \quad 0 \leq \alpha \leq 1, \quad \alpha + \beta = 1.
\]

Hence,

\[
\tilde{X}_t^c = \alpha x^{(a)} + \beta x^{(b)} + \int_0^t \left[ A_t \left( \alpha X_r^{(a)} + \beta X_r^{(b)} \right) + B_t (\alpha v_r^{(a)} + \beta v_r^{(b)}) \right] \, dr + \int_0^t \left[ \sigma_{1,r} + \sigma_{2,r} (|\alpha v_r^{(a)}| + |\beta v_r^{(b)}|) \right] \, dW_r.
\]  

(23)

Note that (23) is not a feasible path. However, by the triangle inequality, we conclude that the diffusion coefficient of $\tilde{X}_t^c$ is larger than the diffusion coefficient of $X_t^c$, in the sense mentioned above. In addition, using the definition of optimal cost-to-go given by (12), we can also conclude for any state path starting at $x^*$ that

\[
J^*(t, x^*) = \inf_{v_t^*} \mathbb{E}^x \left[ \int_t^T f(r, X_r, v_r) \, dr + g(X_T) \right] \leq \mathbb{E}^x \left[ \int_t^T f(r, \tilde{X}_r, v_r) \, dr + g(X_T^c) \right].
\]

(24)

Finally, using the result from Lemma 4 and the convexity of functions $f$ and $g$, we obtain

\[
J^*(t, x^*) \leq \mathbb{E}^x \left[ \int_t^T f(r, \tilde{X}_r, v_r) \, dr + g(X_T^c) \right] \leq \mathbb{E}^x \left[ \int_t^T f(r, X_r, v_r) \, dr + g(X_T^c) \right] + \beta E \mathbb{E}^x \left[ \int_t^T f(r, X_r^{(b)}, v_r^{(b)}) \, dr + g(X_T^{(b)}) \right] = \alpha J^*(t, x^*) + \beta J^*(t, x^*) \]

(24)

which implies that $J^*(t, x^*)$ is convex.

4. OPTIMAL INACTION REGION

4.1 The Dynamic Programming

The DP method is a remarkable mathematical tool to deal with stochastic control problems in continuous case. The following theorem introduces the Hamilton-Jacobi-Bellman (HJB) equation and represents a fundamental result of the method. We drop the arguments
and denote $J^*$, also $J^*_t$, $J^*_v$ and $J^*_{xv}$ stand for the corresponding partial derivative.

**Theorem 6.** (Oksendal, 2007, p 240) Define the optimal cost-to-go function as (12) and let $\mathcal{A}^v$ be the usual generator of an Itô diffusion. Suppose that $J^* \in C^1([0,T]) \times C^2(\mathbb{R}^n)$ satisfies

$$E_t \left[ \int_t^T J_t^* + \mathcal{A}^v J^* (r, X_r) dr + |J^* (T, X_T)| \right] < \infty,$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $T < \infty$, and $v \in \mathcal{U}$. Then,

$$J_t^* + \inf_{v \in \mathcal{U}} H(t, x, v) = 0,$$  

$$J^* (T, x) = g(x),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$; here the Hamiltonian function is $H(t, x, v) := \{ f(t, x, v) + \mathcal{A}^v J^* \}$. Moreover, if a Markovian optimal control $v^*$ exists, the infimum in (25a) is obtained for $v = v^*(t, x)$.

### 4.2 Minimization of the Hamiltonian

The application of the generator $\mathcal{A}^v$ to the CVIU system (4)–(5) evaluated by (6) yields

$$\mathcal{A}^v J^* = (A_v x + B_v v)' J_t^* + \frac{1}{2} \text{tr} \left[ (\sigma_1 v + \sigma_2 v | v) J_{xv}^* (\sigma_1 v + \sigma_2 v | v) \right]$$

$$= x' A_v J_t^* + \frac{1}{2} \Gamma_1 (J_{xv}^*)$$

$$+ \frac{1}{2} \Gamma_2 (J_{xv}^*) v^2 + B_v J^* v + \frac{1}{2} \Gamma_1 (J_{xv}^*) |v|,$$

where $v \in \mathcal{U}$, and $\text{tr} [\cdot]$ denotes the matrix trace. For the last representation, we introduced the notation:

$$\Gamma_i (\Sigma) = \text{trace} \left[ \sigma_{ij} \Sigma \sigma_{ij} \right], \quad i = 1, 2,$$

$$\Gamma_12 (\Sigma) = \text{trace} \left[ \sigma_{ij} \Sigma \sigma_{ij} + \sigma_{ij} \Sigma \sigma_{ij} \right],$$

for $\Sigma \in \mathbb{R}^{n \times n}$ symmetric and positive semidefinite. Note that the absolute value of $v$ appearing at the last term of the equation (26) makes the Hamiltonian not differentiable. On the other hand, the development of nonsmooth analysis provides an interesting way to deal with that.

**Definition 1.** (Clarke, 1987, p 63) Let $h : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $v \in \mathbb{R}^n$. Let $\Phi$ be any set of zero measure in $\mathbb{R}^n$, and let $\Phi_0$ be the set of points in $\mathbb{R}^n$ where $h(v)$ fails to be differentiable. Then, the generalized gradient in $v$, denoted by $\partial_v h(v)$, will be the set

$$\partial_v h(v) = \text{co} \left( \lim_{v_i \to v} \nabla h(v_i) : v_i \notin \Phi, v_i \notin \Phi_h \right),$$

where $\text{co} [\cdot]$ means the closed convex hull.

One can prove that the Hamiltonian is locally Lipschitz in $v$ from the assumption that $J^* \in C^1([0,T]) \times C^2(\mathbb{R}^n)$ has second-order partial derivatives locally limited. Consequently, using Definition 1, we get

$$\partial_v H(t, x, v) = f_v(t, x, v) + F_v (J_{xv}^*) v$$

$$+ B_v J^* + \frac{1}{2} \Gamma_12 (J_{xv}^*) v^2,$$

where $\mathcal{A}^v$ corresponds to the following set

$$\mathcal{A}^v = \begin{cases} +1, & \text{se } v > 0, \\ -1, & \text{se } v < 0, \\ [-1, +1], & \text{se } v = 0. \end{cases}$$

Besides, if we can prove that the Hamiltonian is convex in $v$, it will be possible to determine the sign of $v^*$ based solely on the value of the state $x$. This is presented in more details in Lemma 7. Indeed, the convexity of the optimal cost-to-go $J^* (t, x)$ and the assumption that $f(t, x, v)$ is also convex in $v$ are sufficient to ensure that the Hamiltonian is convex in $v$.

**Lemma 7.** For the optimal control function (7) and the optimal cost-to-go function (12), we have

$$\begin{cases} v^* > 0, & \text{if } x \in \mathcal{B}_1, \\ v^* < 0, & \text{if } x \in \mathcal{B}_2, \\ v^* = 0, & \text{if } x \in \mathcal{B}_3, \end{cases}$$

where

$$\mathcal{B}_1 = \{ x \in \mathbb{R}^n : \lim_{v_i \to 0} \partial_v H(t, x, v) < 0 \},$$

$$\mathcal{B}_2 = \{ x \in \mathbb{R}^n : \lim_{v_i \to 0} \partial_v H(t, x, v) > 0 \},$$

$$\mathcal{B}_3 = \{ x \in \mathbb{R}^n : \mathcal{B}_1 \cup \mathcal{B}_2 \}.$$

**PROOF.** According to (28)–(29), the set $\partial_v H(t, x, v)$ is either a point or a closed connected interval of the line. We also can formulate from the convexity of the Hamiltonian a non-decreasing notion in $v$ for $v \to \partial_v H(t, x, v)$ in the sense that $v_1 \leq v_2 \Leftrightarrow \partial_v H(t, x, v_1) \leq \partial_v H(t, x, v_2), \forall v_1 \in \partial_v H(t, x, v_1)$ and $\forall v_2 \in \partial_v H(t, x, v_2)$.

With this notion at hand, we can determine the sign of $v^*$ observing only $\partial_v H(t, x, v)|_{v=0}$. The analysis yields the following rule:

- **If $\eta < 0$, $\forall \eta \in \partial_v H(t, x, v)|_{v=0}$, then the Hamiltonian is decreasing at $v^* = 0$, and the minimum is attained at the positive half-plane ($v^* > 0$);**
- **If $\eta > 0$, $\forall \eta \in \partial_v H(t, x, v)|_{v=0}$, then the Hamiltonian is increasing at $v^* = 0$, and the minimum is attained at the negative half-plane ($v^* < 0$);**
- **If $\eta = 0 \in \partial_v H(t, x, v)|_{v=0}$, then the optimal control is $v^* = 0$.**

Note in the verification above that we need to check just the sign of the extreme elements of the set $\partial_v H(t, x, v)|_{v=0}$ to determine the sign of $v^*$. Based on the fact that $v \to \partial_v H(t, x, v)|_{v=0}$ is non-decreasing in $v$ and using the definition of the generalized gradient, we have that
lim \( \partial_v H(t,x,v) \) = \( \max_{v \rightarrow 0} \partial_v H(t,x,v) \) \( \left| v = 0 \right. \), (32)
lim \( \partial_v H(t,x,v) \) = \( \min_{v \rightarrow 0} \partial_v H(t,x,v) \) \( \left| v = 0 \right. \), (33)
and it is clear that
\[
\lim_{v \rightarrow 0} \partial_v H(t,x,v) \geq \lim_{v \rightarrow 0} \partial_v H(t,x,v). \quad (34)
\]
From Definition 1, it is noteworthy that the limits above are calculated avoiding sets of zero measure and the points where \( v \rightarrow H(t,x,v) \) fails to be differentiable. From these arguments, we obtain
\[
\lim_{v \rightarrow 0} \partial_v H(t,x,v) < 0 \Rightarrow v^\ast > 0, \quad \text{(Region } \mathcal{R}_1) \]
\[
\lim_{v \rightarrow 0} \partial_v H(t,x,v) > 0 \Rightarrow v^\ast < 0, \quad \text{(Region } \mathcal{R}_2) \]
and we can guarantee that the regions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) will not overlap. As a consequence, we can identify three complementary regions covering the state space: \( \mathcal{R}_1, \mathcal{R}_2 \) and \( \mathcal{R}_3 \). They are given by (31) defined in the statement of the Lemma. The region \( \mathcal{R}_3 \) is called Region of Inaction, and it is associated with \( v^\ast = 0 \).

4.3 When is the region of inaction nondegenerate?

One could suppose that the regions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) may be large enough to fill the whole state space. In this case, the region \( \mathcal{R}_3 \) could be just an hyperplane on the state space and the equation (34) would be just an equality for all \( (t,x) \in [0,T] \times \mathcal{R}_3 \). In order to prove that the region of inaction \( \mathcal{R}_3 \) exists as a nonempty hypersurface, it is sufficient to prove that the inequality (34) holds strictly for all \( (t,x) \in [0,T] \times \mathbb{R}^n \). Due to its importance, this result will be presented as a lemma.

Lemma 8. Let \( t \rightarrow X_t \) be the controlled diffusion process defined in (4)–(5) with \( \sigma_{ij} = \lambda_i \sigma_{2_j}, \lambda_i > 0 \), and suppose that the optimal cost-to-go function \( J^\ast(t,x) \) is not an hyperplane in \( \mathbb{R}^n \). Then the region \( \mathcal{R}_3 \) defined in (31c) is a nonempty hypersurface on the state space.

**PROOF.** Consider the problem stated in Section 2 and the result from Lemma 7. We can better represent the region \( \mathcal{R}_3 \) defined in (31c) as the following set
\[
\mathcal{R}_3 = \left\{ x \in \mathbb{R}^n : \lim_{v \rightarrow 0} \partial_v H(t,x,v) < 0 < \lim_{v \rightarrow 0} \partial_v H(t,x,v) \right\}
= \left\{ x \in \mathbb{R}^n : 0 < \partial_v H(t,x,v) \left| v = 0 \right. \right\}. \quad (35)
\]
Developing the limits in (34), we obtain
\[
\lim_{v \rightarrow 0} [f_v(t,x,v)] + B'_v J^\ast + \frac{1}{2} \Gamma_{12} (J^\ast)_x \geq 0
\]
\[
\lim_{v \rightarrow 0} [f_v(t,x,v)] + B'_v J^\ast - \frac{1}{2} \Gamma_{12} (J^\ast)_x. \quad (36)
\]
Hence, by the assumption that \( f(t,x,v) \) is continuously differentiable in \( v \), the inequality (36) will be strict for all \( (t,x) \in [0,T] \times \mathbb{R}^n \) if, and only if, \( \Gamma_{12} (J^\ast)_x > 0 \). Since \( J^\ast \) is convex in \( x \) (see Theorem 5) and, consequently, its Hessian is positive semidefinite (Boyd and Vandenberge, 2004, p. 71), it follows from the assumption \( \sigma_{ij} = \lambda_i \sigma_{2_j}, (\lambda_i > 0) \) that
\[
J^\ast_x \geq 0 \Rightarrow \Gamma_{12} (J^\ast)_x > 0. \quad (37)
\]
Finally, the inequality (37) is strict when at least one element on the main diagonal of the Hessian is not equal to zero. Thus, by the assumption that \( J^\ast(t,x) \) is not an hyperplane in \( \mathbb{R}^n \), we can exclude the case in which all the elements on the main diagonal of the Hessian is equal to zero, completing the proof.

5. CONCLUSIONS

In this paper, we developed a theoretical framework for continuous-time CVIU systems and we presented the DP method to deal with the associated stochastic control problem. The convexity of the cost-to-go function was shown and it makes possible to understand some important features of the solution. It is remarkable that from the control optimization of the CVIU system, it appears a region of “inaction”, in which the optimal solution is to keep the control unchanged. This behavior is in accordance with the idea of acting on an uncertain system.

6. REFERENCES


