Asymptotic stability of linear stochastic systems with delay driven by a Bernoulli process *

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Abstract: This note characterizes the asymptotic stability in the mean for a class of linear stochastic systems. The system is subject to random delays driven by a Bernoulli process. The approach is quite general and can be used to handle systems subject to failures, such as the ones that may suffer from packet dropouts in transmission lines. A real-time application is presented to illustrate the results.

Keywords: discrete-time systems, stability of stochastic systems, systems with delay, Bernoulli process.

1. INTRODUCTION

Systems subject to delay and failures on the communication links are quite common in practice, and much effort has been made in the last years to derive new results for such a class of systems, see for instance Mahmoud [2010], Verriest and Michiels [2009], Hespanha and Naghshtabrizi [2007], Xia et al. [2009] for an account. Our approach contributes towards this direction by considering the special case of linear systems with delay depending on a particular representation of failures and successes, and driven by a Bernoulli process.

Let us now introduce the system under consideration. With \((\Omega, \mathcal{F}, P)\) as a fixed probability space, let us consider the system

\[
x(k + 1) = Ax(k) + Bx(k - \delta_k) + Dw(k), \quad \forall k \geq 0,
\]

with \(x_0 \in \mathbb{R}^n\) and \(\delta_0 = 0\), where \(x_k\) and \(w_k, k \geq 0\) are processes taking values respectively in \(\mathbb{R}^r\) and \(\mathbb{R}^s\), and \(A, B\), and \(D\) are matrices with compatible dimensions. The noisy input \(\{w_k\}\) forms an independent and identically distributed (iid) process with zero mean and covariance matrix equals to the identity for every \(k \geq 0\). Associated with (1), we consider an usual Bernoulli random variable with probability of successes and failures given by \(p\) and \(q := 1 - p\), taking values “0” and “1”, respectively.

The time instants

\[0 < k_1 < k_2 < \ldots < k_i < \ldots\]

denote the time in which the information is received with success by the system (1). Indeed, it marks each instant for which the delay process resets to zero, i.e., \(\delta_{k_i} = 0, i \geq 0\), see Fig. 1. One can also see that the random variable \(\delta_k\) counts the number of sequential occurrences of failures and resets to zero whenever a success happens. Note that the delay process follows the rule

\[\delta_k = k - k_i, \quad k = k_i, \ldots, k_{i+1} - 1.\]

The main contribution of this paper is to characterize the asymptotic stability in the mean for the linear stochastic system (1) subject to delay (see Remark 6 for a formal definition). Actually, we prove that

\[\|E|x(k)|\| \rightarrow 0 \quad \text{almost surely,}\]

if the following two mild conditions are simultaneously satisfied:

(1) The spectral radius of \(qA\) is strictly less than one.

(2) The spectral radius of \(H(q)\) is strictly less than one, where

\[H(q) = (1 - q)A[I + qA(I - qA)^{-1}] + [I + qA(I - qA)^{-1}]B.\]

The result of this paper is of particular interest for networked control systems with packet dropouts, because the delay representation is quite general and can be used to incorporate failures in the communication links. This is illustrated in the paper by a real application that controls the water level in a tank process through an emulated network, see Section 3.
The real numbers are denoted by \( \mathbb{R} \). The symbol \( \| \cdot \| \) stands for the Euclidean norm in \( \mathbb{R}^n \). Let \( \sigma(\cdot) \) be the spectral radius operator. The indicator function of the set \( C \) is represented by \( \mathbb{1}_C \).

Let us recall the delay structure in the system (1). The identity in (2) enables us to rewrite (1) equivalently as
\[
\begin{align*}
\delta_k &= Ax(k) + Bx(k_i) + Dw(k_i) \\
\delta_{k+1} &= Ax(k+1) + Bx(k_{i+1}) + Dw(k_{i+1})
\end{align*}
\]
Hence,
\[
x(k+1) = Ax(k) + Bx(k_i) + Dw(k_i)
\]
\[
x(k_{i+1}) = Ax(k_{i+1}) + Bx(k_i) + Dw(k_{i+1})
\]
or equivalently, for each \( s \in \{1, \ldots, k+i+1-k\} \),
\[
x(s) = A^s + \sum_{j=0}^{s-1} A^j B x(k_j) + \sum_{j=0}^{s-1} A^{s-j} Dw(k_j)
\]
\[
x(k+s) = A^s + \sum_{j=0}^{s-1} A^j B x(k_j) + \sum_{j=0}^{s-1} A^{s-j} Dw(k_j).
\]

\[
M(s) = A^s + \sum_{j=0}^{s-1} A^j B, \quad s = 1, 2, \ldots
\]
with \( M(0) = I \) and
\[
\nu(s, k) = \sum_{j=0}^{s-1} A^{s-j} Dw(k_j), \quad k, s = 1, 2, \ldots
\]
with \( \nu(0, k) = 0 \) for each \( k \geq 0 \).

The process \( \{\delta_k\} \) is driven, by assumption, by an iid Bernoulli process, hence the interarrival times defined as
\[
s_i = k_{i+1} - k_i, \quad \forall i \geq 0,
\]
follow the geometric distribution [Bertsekas and Tsitsiklis, 2002, Sec. 5.1]
\[
\Pr(s_i = n) = pq^{n-1}, \quad \forall i \geq 0, \forall n \geq 1.
\]

Combining (4)-(7), we have
\[
x(k+1) = M(s_i) x(k_i) + \nu(s_i, k_i), \quad \forall i \geq 0.
\]
It is noteworthy that both \( s_i \) and \( k_i \) are independent random variables [Bertsekas and Tsitsiklis, 2002, Sec. 5.1]. Thus, the identity
\[
E[M(s_i) x(k_i)] = E[M(s_i) x(k_i)] + E[\nu(s_i, k_i)], \quad \forall i \geq 0.
\]

\[\text{Remark 1.}\] We state that
\[
E[\nu(s_i, k_i)] = 0, \quad i = 0, 1, \ldots
\]
Indeed, by assumption, the iid process \( \{s_i\} \) is independent of the noise process \( \{w(k_i)\} \), we can write
\[
E\left[ \sum_{j=0}^{s_i-1} A^{s-j} Dw(k_i + j) \right] = E\left[ \sum_{j=0}^{s_i-1} A^{s-j} D \right] E\left[ w(k_i + j) \right].
\]
This identity and the other assumption that \( E[w_k] \equiv 0 \) yield the result in (11).

The next result defines a deterministic matrix, useful for the stability analysis.

\[\text{Lemma 2.}\] If \( \sigma(qA) < 1 \), then there holds
\[
E[M(s_i)] = H(q), \quad i = 0, 1, \ldots
\]
where
\[
H(q) = (1-q) A[I + qA(I-qA)^{-1}] + [I + qA(I-qA)^{-1}] B.
\]

\[\text{Proof.}\] By definition,
\[
E[M(s_i)] = \sum_{s=1}^{\infty} M(s) \Pr(s_i = s).
\]
Using (5) and (8) in the above identity, we have
\[
E[M(s_i)] = \sum_{s=1}^{\infty} M(s) q^{s-1} (1-q)
\]
\[
= (1-q) \sum_{s=0}^{\infty} q^s \left( A^{s+1} + \sum_{j=0}^{s} A^j B \right).
\]

A simple rearrangement in the terms of (14) shows that \( E[M(s_i)] \) is equal to
\[
(1-q) \left( A + B \right) + A \sum_{s=1}^{\infty} (qA)^s + \sum_{s=1}^{\infty} \sum_{j=0}^{s} q^s A^j B.
\]

Since, by assumption, \( \sigma(qA) < 1 \), we have The second term inside the brackets of (15) equals
\[
A \sum_{s=1}^{\infty} (qA)^s = qA^2 (I - qA)^{-1}.
\]
In addition,
\[ \sum_{s=1}^{\infty} q^s A^j B = q(I + A)B + q^2(I + A + A^2)B + \ldots + q^s(I + A + A^2 + \ldots + A^j)B + \ldots \]
\[ = q(1 + q + q^2 + \ldots + q^s)(A + A^2 + A^3 + \ldots)B + \ldots \]
\[ = \frac{1}{1-q} (qI + qA + (qA)^2 + (qA)^3 + \ldots)B. \]
Thus,
\[ \sum_{s=1}^{\infty} q^s A^j B = \frac{1}{1-q} (qI + qA(I - qA)^{-1})B. \]

These facts enable us to conclude that
\[ E[M(s_i)] = (1-q)(A+B) + (1-q)qA^2(I-qA)^{-1}B \]
\[ + qB + qA(I-qA)^{-1}B = H(q), \]
which shows the result. \(\square\)

As we shall see, the spectral radius of the matrix \(H(q)\) will be decisive to characterize the asymptotic stability in mean for (1).

The next result reinforces the relevance of the matrix \(H(q)\). The proof of the next result follows immediately from Lemma 2, connected with (9) and (10).

**Lemma 3.** If \(\sigma(qA) < 1\), then
\[ E[x(k_{i+1})] = H(q)E[x(k_i)], \quad \forall i \geq 0, \]
where \(H(q)\) satisfies (13).

The next result is a direct application of Lemma 3.

**Theorem 4.** Suppose that \(\sigma(qA) < 1\) holds. Then the two assertions below are equivalent.

(i) There exist two constants \(\alpha > 0\) and \(0 < \beta < 1\) such that
\[ \|E[x(k_i)]\| \leq \alpha \beta^i \|x_0\|, \quad \forall i \geq 0. \]

(ii) The matrix \(H(q)\) in (13) is stable, i.e., \(\sigma(H(q)) < 1\).

Theorem 4 presents an interesting condition, based on the spectral radius of the matrix \(H(q)\), to assert the exponential decay of the sequence \(\{E[x(k_i)]\}\). This property is the key to characterize the asymptotic stability in the mean for system (1).

### 2.1 Asymptotic stability in the mean

Let us introduce the following stability concepts for the system (1).

**Definition 5.** ([Arnold, 1974, p.188], Vargas and do Val [2010])
The stochastic system (1) is called

(i) **stable in the mean** if for any given \(k_0 \geq 0\) and \(\epsilon > 0\), there exists a positive number \(\delta(\epsilon, k_0)\) such that \(\|x_0\| < \delta\) and \(x(k_0) = x_0\), then \(\|E[x(k)]\| \leq \epsilon\) holds for all \(k \geq k_0\).

(ii) **asymptotically stable in the mean** if for any \(k_0 \geq 0\), there exists a positive number \(\delta(k_0)\) such that \(\|x_0\| < \delta\) and \(x(k_0) = x_0\), then for any \(\epsilon > 0\), there exists a natural number \(T(\epsilon, \delta, k_0)\) such that \(\|E[x(k)]\| < \epsilon\) for all \(k > k_0 + T\).

**Remark 6.** It is immediate from Definition 5 that asymptotic stability in the mean is equivalent to the condition \(\|E[x_k]\| \to 0\) as \(k \to \infty\). Notice that asymptotic stability in the mean suffices for stability in the mean. \(\square\)

Now we revisit the well-known Borel-Cantelli lemma.

**Lemma 7.** (Borel-Cantelli lemma [Çınlar, 2011, p. 98], [Davis and Vinter, 1985, p. 29]). Let \(\{S_n\}\) be a sequence of events. Then
\[ \sum_{n=0}^{\infty} \Pr(S_n) < \infty \Rightarrow \sum_{n=0}^{\infty} \mathbb{I}_{S_n} < \infty \text{ almost surely.} \]

**Remark 8.** A consequence of the Borel-Cantelli lemma is that, in the case that \(\sum_{n=0}^{\infty} \Pr(S_n) < \infty\), then with probability one (or almost surely) there exists a number \(N > 0\) sufficiently large for which \(S_n\) does not occur when \(n \geq N\). In our setup, this result can be interpreted as follows. Let us fix the arbitrary \(i_o\)-th interarrival time in order to link the random variable \(s_{i_o} = k_{i+1} - k_{i_o}\) with the specific sequence of time events \(S_n = \{s_{i_o} = n + 1\}, \quad n = 0, 1, \ldots\)

It follows from (8) that
\[ \sum_{n=0}^{\infty} \Pr(S_n) = \sum_{n=0}^{\infty} q^n(1-q) = 1. \]

Applying the Borel-Cantelli lemma, we assure the existence of a constant \(N > 0\) such that \(s_n < N\) almost surely. Since \(i_o\) was taken arbitrarily, we can conclude that \(s_i = k_{i+1} - k_i < N, \quad \forall i \geq 0, \quad \text{almost surely.} \]

Now we are able to present the main result of this paper.

**Theorem 9.** If \(\sigma(qA) < 1\) and \(\sigma(H(q)) < 1\), then the stochastic system (1) is asymptotically stable in the mean almost surely.

**Proof.** In this proof we want to show that
\[ n = 0, \ldots, N \Rightarrow E[x(k_i + n)] \to 0 \quad \text{as} \quad i \to \infty, \]
where \(N > 0\) represents the constant that satisfies (17). The result in (18) is prerequisite to guarantee that
\[ \lim_{k \to \infty} E[x(k)] = 0 \quad \text{almost surely.} \]

Indeed, it is straightforward to verify that the arbitrary \(k\)-th instant belongs to some interval \([k_i, k_{i+1})\), and this one is bounded by \(N > 0\) (see (17)). This partition of time and (18) assure that (19) holds.

The remaining part of this proof is dedicated to show (18). To begin with, note from Theorem 4 that \(E[x(k_i)] \to 0\) as \(i \to \infty\) and this proves (18) for \(n = 0\).

Now, with \(0 < n < N\), let us define the quantities \(\tilde{j}_i\) and \(\tilde{r}_i\) as follows.
\[ \tilde{j}_i = \max\{j \in \mathbb{N} : k_{i+j} \leq k_i + n\}, \quad \text{and} \quad \tilde{r}_i = k_i + n - k_{i+j}. \]
It is immediate from definition that \(0 \leq \tilde{j}_i \leq n\) and \(0 \leq \tilde{r}_i \leq n\) for all \(i \geq 0\). Moreover, we can use \(\tilde{j}_i\) and \(\tilde{r}_i\) into (3) to write down the recurrence...
\[ x(k_{i+j_i}) + 1 = Ax(k_{i+j_i}) + Bx(k_{i+j_i}) + Dw(k_{i+j_i}) \]
\[ x(k_{i+j_i} + 2) = Ax(k_{i+j_i}) + Bx(k_{i+j_i}) + Dw(k_{i+j_i}) \]
\[ x(k_{i+j_i} + i) = Ax(k_{i+j_i} + i - 1) + Bx(k_{i+j_i} + i - 1) + Dw(k_{i+j_i} + i - 1). \]

Taking the expected value operator in the first recurrence, and recalling that \( E[x(k_{i+1})] = 0 \), we have
\[ E[x(k_{i+j_i})] = (A + B) E[x(k_{i+j_i})]. \]

On the other hand, due to the fact that \( \{k_{i+j_i}\} \) is a subsequence of \( \{k_i\} \), and that \( E[x(k_i)] \to 0 \) as \( i \to \infty \), we can conclude that
\[ \lim_{i \to \infty} E[x(k_{i+j_i})] = 0. \]

Again, we can repeat the argument and apply the expected value operator in the second recurrence to obtain
\[ E[x(k_{i+j_i} + 2)] = AE[x(k_{i+j_i} + 1)] + BE[x(k_{i+j_i})], \]

which in turn combined with (21) and (22) produces
\[ \lim_{i \to \infty} E[x(k_{i+j_i} + 2)] = 0. \]

By proceeding further in the argument, one obtains
\[ \lim_{i \to \infty} E[x(k_{i+j_i} + t)] = 0, \quad t = 3, \ldots, \bar{r}_i. \]

The result in (18) now follows because, if we let \( t = \bar{r}_i \) in (23), and recalling that
\[ E[x(k_{i+j_i} + \bar{r}_i)] = E[x(k_i + n)], \]
then we get
\[ \lim_{i \to \infty} E[x(k_i + n)] = 0. \]

This argument completes the proof. \( \square \)

3. APPLICATION: STABILITY OF A WATER LEVEL PROCESS WITH PACKET DROPOUTS

Data transmitted across communication networks is, in most of the time, subject to failures in the communication process. Commonly, the data is enclosed into packets and diverse events may cause a packet dropout. In a situation of a loss of packet, it is extremely desired to assure the stability of the overall system that is operated via a network, see Hespanha and Naghshtabrizi [2007], Zhang and Yu [2007], Sun and Qin [2011], Ishido et al. [2011].

To contribute towards the knowledge of stability of communication networks, we present a real application of a process controlled via network links. The idea is that of controlling the level of water in a tank using transmission lines to implement the closed-loop feedback links, see Fig. 2 for a pictorial representation. The transmission lines are subject to packet dropouts, and we verify the stability of the process under this scenario.

The experimental testbed is based on the Level and Temperature Control Module 2325, made up by Datapool Eletronica Ltda, Brazil, associated with a National Instruments USB-6008 data acquisition card to perform a physical link with the computer, see Fig. 3. A network is emulated within the Matlab environment to implement the feedback links, observing that these ones are subject to the Bernoulli variable with a fixed probability of failures \( q \). The computer is also used to implement physically the discrete-time version of the phase-lag controller [Phillips and Harbor, 1996, Sec. 9.4]. The sampling time is 35 milliseconds.

The stochastic linear system representing the water level process is given by
\[ x(k+1) = Ax(k) + By(k) + D\omega(k), \]
\[ y(k) = Cx(k - \delta_k), \quad k \geq 0, \]

where \( \{y(k)\} \) represents the system output, and the system parameters are
\[ A = \begin{bmatrix} 1.996439 & -0.996454 \\ 0.999996 & -3.518042 \cdot 10^{-6} \end{bmatrix}, \]
\[ B = 10^{-3} \begin{bmatrix} 1.878949 \\ 1.876749 \end{bmatrix}, \]
\[ C = 10^{-3} \begin{bmatrix} 1.876762 \\ 1.874540 \end{bmatrix}, \quad D = 0.05I. \]

The water level in the tank, measured in centimeters, is given by
\[ h(k) = 4y(k) + 11.64, \quad \forall k \geq 0. \]

From the model of (24), one can easily verify that the spectral radius of the matrix \( H(q) \) as in (13) is strictly lower than one for all \( 0 < q < 1 \), see Fig. 4. Theorem 9 then confirms that the system (24) is asymptotically stable in the mean almost surely, i.e,
Fig. 4. Spectral radius of the matrix $H(q)$ as in (13), for each $0 < q < 1$, for the water level process of Section 3.

$$\lim_{k \to \infty} \|E[x(k)]\| = 0 \quad \text{almost surely.}$$

Now, the main objective is to verify the behaviour of the water level in the tank when the data transmission is subject to a high and low probability of failures. It is clear that these will influence the failures in the data transmitted through the emulated network, and to clarify this we carry out in practice two distinct realizations taking the probability of failures equals to $q = 0.99$ and $q = 0.01$. The experimental results obtained in practice are depicted in Fig. 5. Note that the water level response in the former case is poorer when compared with the latter one, and this fact is reasonable due to the heavy occurrence of failures that turn the control of the pump less efficient.

In summary, even in the real scenario of failures in the data transmission, the equipment maintains a stable behaviour as predicted by the result of Theorem 9.

REFERENCES


