ROLLING A PSEUDOHYPERBOLIC SPACE OVER THE AFFINE TANGENT SPACE AT A POINT

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Abstract: This paper describes how a pseudohyperbolic space rolls, without slipping or twisting, over its affine tangent space at a point. These two manifolds have the same dimension and are considered embedded in the same pseudo-Euclidean space. After defining rolling maps, we derive the kinematic equations for these rolling motions and present explicit solutions when the manifolds roll along geodesic curves. The kinematic equations can be seen as a control system evolving on a certain Lie group and we prove that this system is controllable. Copyright CONTROLO2012.

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1. INTRODUCTION

The most classical of all rolling systems is that of a 2-sphere rolling on its tangent plane at a point. This fits in a more general situation in which one manifold rolls, without slipping or twisting, over another manifold of the same dimension, both embedded in the same Euclidean space. The geometry of these rolling motions has been studied in (Sharpe, 1997), where a formal definition of rolling map is given and became the basis for interesting properties and further developments. These rolling systems have nonholonomic constraints which correspond to the requirement that slipping or twisting is not allowed. Some particular situations where the kinematic equations of rolling have been derived appear, for instance, in (Zimmerman, 2005) and (Hüper and Leite, 2007). In the latter, the geometry of rolling is successfully applied in the implementation of an efficient algorithm to generate smooth interpolating curves on manifolds. The work of (Sharpe, 1997) has been recently generalized in (Hüper et al., 2011) to accommodate rolling motions inside general Riemannian manifolds. The interest in rolling pseudo-Riemannian manifolds started with (Jurdjevic and Zimmerman, 2008) with the hyperbolic sphere and has been extended to Lorentzian spheres in (Korolkov and Leite, 2011) and to quadratic Lie groups in (Leite and Crouch, 2010). The present paper is also devoted to rolling motions of certain manifolds embedded in pseudo-Riemannian manifolds. More precisely, we study rolling motions of pseudo-hyperbolic spaces over the affine tangent space at a point, both embedded in a pseudo-Euclidean space.

The organization of the paper is as follows. We start with some preliminaries about pseudo-Riemannian manifolds, in particular, with the geometry of pseudo-orthogonal groups, which play a very important role in this context, since the action of these groups is essential to describe rolling motions. In section 4 we present the rolling manifolds and define all the necessary ingredients that will be used in Section 5 to derive the kinematic equations of rolling. In this

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section we also present important cases when the kinematic equations can be solved in closed form. This situation corresponds to rolling along geodesic curves. Finally, in the last section, we rewrite the kinematic equations as a left-invariant control system evolving on a particular Lie group, and prove the controllability property.

2. PSEUDO-ORTHOGONAL GROUPS

We start with the basic background about the pseudo-Riemannian (also called semi-Riemannian) manifolds that will appear throughout the paper. For more details, we refer to (O’Neill, 1983).

The tangent space of $\mathbb{R}^n$ at a point $p$, $T_p\mathbb{R}^n$, is identified with the vector space $\mathbb{R}^n$, resulting from the canonical isomorphism which, in terms of natural coordinates $x_1, \cdots, x_n$, sends $v_p = v_1 \frac{\partial}{\partial x_1} \big|_p + \cdots + v_n \frac{\partial}{\partial x_n} \big|_p \in T_p(\mathbb{R}^n)$ to $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$.

Now, equip the differentiable manifold $\mathbb{R}^n$ with a pseudo-Riemannian metric defined on each $T_p(\mathbb{R}^n)$ as

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{i=\nu+1}^{n} v_i w_i. \quad (1)$$

In spite of this notation, which is usually used for the Euclidean metric, (1) only defines the Euclidean (Riemannian) metric when $\nu = 0$. Hereafter, and for $0 \leq \nu \leq n$, we denote this pseudo-Riemannian manifold by $\mathbb{R}_{\nu}^n$ (this is sometimes also called pseudo-Euclidean space). If $M$ is a pseudo-Riemannian submanifold of $\mathbb{R}_{\nu}^n$ and $p \in M$, the following direct sum decomposition holds:

$$T_p \mathbb{R}_{\nu}^n = T_p M \oplus T_p M^\perp. \quad (2)$$

Consider the diagonal matrix $J_{\nu} = (\delta_{ij} \varepsilon_i)$ whose main diagonal entries are $\varepsilon_1 = \cdots = \varepsilon_\nu = -1, \varepsilon_{\nu+1} = \cdots = \varepsilon_n = 1$. Associated with this matrix one can define a matrix Lie group (closed subgroup of $GL(n, \mathbb{R})$)

$$O_{\nu}(n) := \{ R \in GL(n, \mathbb{R}) : R^\top J_{\nu} R = J_{\nu} \},$$

which turns out to be a pseudo-Riemannian manifold known as pseudo-orthogonal group.

$$SO_{\nu}(n) := \{ R \in O_{\nu}(n) : \det(R) = 1 \}.$$

is a subgroup of $O_{\nu}(n)$, known as the special pseudo-orthogonal group. When $\nu$ is equal to 0 or $n$, the group $O_{\nu}(n)$ is the Riemannian orthogonal group $O(n)$ of the linear isometries of the Euclidean space $\mathbb{R}^n$, and $SO_{\nu}(n)$ is connected. When $0 < \nu < n$, each matrix in $O_{\nu}(n)$ can be decomposed in the form

$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \quad (2)$$

where $R_1$ and $R_4$ are invertible matrices of order $\nu$ and $(n-\nu)$ respectively. It turns out that $O_{\nu}(n)$ splits into four disjoint sets indexed by the signs of $\det(R_1)$ $\in \det(R_4)$ (in this order). The connected component containing the identity matrix, hereafter denoted by $SO_{\nu}^+(n)$, is defined by

$$SO_{\nu}^+(n) := \{ R \in SO_{\nu}(n) : \det(R_1) > 0, \det(R_4) > 0 \}.$$

This Lie subgroup plays an important role in the context of rolling. As will become clear, the rolling motion of a submanifold embedded in $\mathbb{R}_{\nu}^n$ can be described by certain isometries of the embedding space.

Each isometry of $\mathbb{R}_{\nu}^n$, for $0 \leq \nu \leq n$, has a unique expression of the form

$$T_{x} R : \mathbb{R}_{\nu}^n \mapsto \mathbb{R}_{\nu}^n \quad p \mapsto Rp + s$$

with $R \in O_{\nu}(n)$ and $s \in \mathbb{R}^\nu$ and the composition of two isometries satisfies

$$(T_{s_2} T_{s_1})(T_{x_1}, R_1) = T_{s_2 + s_1} R_2 R_1.$$

Geometrically, $T_{x} R$ acts on $\mathbb{R}_{\nu}^n$ by pseudo-orthogonal transformations $R$ and translations $s$. For this reason, one can consider the natural identification of the isometries of $\mathbb{R}_{\nu}^n$ with the group

$$O_{\nu}(n) \times \mathbb{R}^\nu := \{ (R, s) : R \in O_{\nu}(n), s \in \mathbb{R}^\nu \},$$

having group operations defined by

$$(R_2, s_2) * (R_1, s_1) := (R_2 R_1, R_2 s_1 + s_2) \quad (R, s)^{-1} := (R^{-1}, -R^{-1} s)$$

and identity element $(I, 0)$.

However, in order to preserve orientations we work with the connected Lie subgroup of $O_{\nu}(n) \times \mathbb{R}^\nu$

$$SO_{\nu}^+(n) \times \mathbb{R}^\nu := \{ X = (R, s) : R \in SO_{\nu}^+(n), s \in \mathbb{R}^\nu \}.$$

For convenience, we identify both $SO_{\nu}^+(n)$ and $SO_{\nu}(n)$ with the rotation group $SO(n)$. The Lie algebra of $SO_{\nu}(n)$ is defined as

$$so_{\nu}(n) := \{ S : S^\top = -J_{\nu} S J_{\nu} \},$$

while the Lie algebra of $SO_{\nu}^+(n) \times \mathbb{R}^\nu$ is $so_{\nu}(n) \oplus \mathbb{R}^\nu$. The matrices in $so_{\nu}(n)$ may be partitioned as

$$S = \begin{bmatrix} A_1 & B^\top \\ B & A_2 \end{bmatrix},$$

where $A_1$ and $A_2$ are skew-symmetric matrices of order $\nu$ and $n-\nu$, respectively, and $B$ is any $(n-\nu) \times \nu$ matrix.

3. DEFINITION OF ROLLING FOR SUBMANIFOLDS OF $\mathbb{R}_{\nu}^n$

The classical definition of rolling, as given in (Sharpe, 1997) for rolling motions inside an Euclidean space, has been generalized in (Hüper et al., 2011) for the situation when the ambient space is a general Riemannian manifold, and in (Leite and Crouch, 2010) for the pseudo-Riemannian case. A particular situation of the later, for the Lorentzian sphere rolling on the affine space at a point appeared in (Korolko and
Leite, 2011). Another example treated in (Jurdjevic and Zimmerman, 2008), and in more generality in (Leite and Crouch, 2010), is the case of the hyperbolic sphere rolling on the affine space at a point. These examples correspond to embeddings in the pseudo-Euclidean space \( R^m_0 \). The present work generalizes the results in (Korolkov and Leite, 2011), by considering rolling motions of hypersurfaces embedded in \( R^m_0 \), for any index \( n \leq \nu \leq m \).

Let \( G \) be the connected Lie subgroup of the group of isometries of \( R^m_0 \) that preserve orientation, that is,

\[
G := SO^+_{\nu}(n) \ltimes R^m_0.
\]

The action, denoted by \( \circ \), of \( G \) on \( R^m_0 \) is defined in the usual manner through

\[
X \circ p := Rp + s, \quad \forall X = (R, s) \in G, \quad \forall p \in R^m_0.
\]

**Definition 3.1.** Let \( M \) and \( N \) be two pseudo-Riemannian submanifolds of \( R^m_0 \) and \( \alpha : [0, \tau] \rightarrow M \) a smooth curve in \( M \). A rolling map of \( M \) on \( N \), without slipping or twisting, along the curve \( \alpha \), is a smooth curve \( X = (R, s) : [0, \tau] \rightarrow G \) in \( G \) that satisfies the following three conditions (1), (2) and (3), for each \( t \in [0, \tau] \).

1. **Rolling conditions:**
   (a) \( X(t) \circ \alpha(t) = R(t)\alpha(t) + s(t) \in N \),
   (b) \( TX(t)\alpha(t)(X(t) \circ M) = TX(t)\alpha(t)N \).

   The curve \( \alpha \) is called the rolling curve, while the curve in \( N \) defined by \( \alpha_{dev}(t) = X(t) \circ \alpha(t) \) is called the development of \( \alpha \) on \( N \).

2. **No-slip condition:**
   \[
   \dot{s}(t) = \dot{R}(t)R^{-1}(t)(s(t) - \alpha_{dev}(t)).
   \]

3. **No-twist conditions:**
   (a) tangential part
   \[
   R(t)\dot{R}(t)(\Omega(t) \in (T_{\alpha_{dev}})N), \quad \forall \Omega \in T_{\alpha_{dev}}N;
   \]
   (b) normal part
   \[
   \dot{R}(t)R^{-1}(t)\xi \in T_{\alpha_{dev}}N, \quad \forall \xi \in (T_{\alpha_{dev}}N)^\perp.
   \]

This definition can be extended to the situation when \( \alpha \) in only piecewise smooth. In this case \( X \) is also piecewise smooth and the constraints (2) and (3) are valid for almost all \( t \).

4. DEFINING THE ROLLING MANIFOLDS

Our objective is to describe the rolling motion of a hyper-quadratic on the affine tangent space at a point, when both are embedded in a pseudo-Euclidean space. This section is devoted to introduce the rolling manifolds and some useful properties.

Let \( n \geq 1, 0 \leq \nu \leq n \) and \( r > 0 \). The pseudo-hyperbolic space in \( R^m_{\nu+1} \) is the hyper-quadratic, with index \( \nu \), dimension \( n \) (and radius \( r \)), defined by

\[
H^\nu_r := \{ p \in R^m_{\nu+1} : \langle p, p \rangle = -r^2 \}.
\]

The pseudo-metric is defined by

\[
\langle v_1, v_2 \rangle = v^T J \nu v_2,
\]

with \( J_{\nu+1} = \text{diag}(-I_{\nu+1}, I_{n-\nu}) \).

\( H^\nu_r \) is connected whenever \( \nu \geq 1 \), but for \( \nu = 0 \) it has two connected components, the upper sheet which contains \((r, 0, \cdots, 0)\), and the lower sheet which contains \((-r, 0, \cdots, 0)\). Nevertheless, these two components may be identified projectively and for our proposes here we consider that \( H^\nu_r \) represents the upper sheet only, that is,

\[
H^\nu_r = \{ p \in R^m_{\nu+1} : \langle p, p \rangle = -r^2, p_1 > 0 \},
\]

where \( p_1 \) is the first component of \( p \). The following results, can be easily derived.

**Proposition 4.1.**

- \( SO^{+}_{\nu+1}(n+1) \) keeps \( H^\nu_r \) (\( \nu \) in-viant.
- \( SO^{+}_{\nu+1}(n+1) \) acts transitively in \( H^\nu_r \) and, consequently, any curve in \( H^\nu_r \) starting at a point \( p_0 \) is of the form \( R(t)p_0 \), for \( t \rightarrow R(t) \) a curve in \( SO^{+}_{\nu+1}(n+1) \).
- \( \forall \Omega \in so_{\nu+1}(n+1), \quad \forall t \in R, \quad e^{t\Omega} \in SO^{+}_{\nu+1}(n+1) \).
- If \( t \rightarrow R(t) \) is a smooth curve in \( SO^{+}_{\nu+1}(n+1) \) then \( \dot{R}(t)R^{-1}(t) \) and \( R^{-1}(t)\dot{R}(t) \) belong to \( so_{\nu+1}(n+1) \).

The following proposition characterizes the tangent space of \( H^\nu_r \) at \( p_0 \), the corresponding affine space containing \( p_0 \), and the normal space at \( p_0 \).

**Proposition 4.2.**

If \( p_0 \) is any point in \( H^\nu_r \), then

\[
T_{p_0}H^\nu_r = \{ v \in R^m_{\nu+1} : v = \Omega p_0, \Omega \in so_{\nu+1}(n+1) \};
\]

\[
T^\nu_{p_0}H^\nu_r = \{ p_0 + \Omega p_0, \Omega \in so_{\nu+1}(n+1) \};
\]

\[
\langle T_{p_0}H^\nu_r \rangle = T_{p_0}R p_0.
\]

**Proof.** Let \( V = \{ \Omega p_0 : \Omega \in so_{\nu+1}(n+1) \} \). Clearly \( V \subseteq T_{p_0}H^\nu_r \), since if we take any \( \nu \in \Omega \in V \) then the curve \( \gamma(t) = e^{t\Omega}p_0 \in H^\nu_r \) satisfies \( \gamma(0) = p_0 \) and \( \gamma(0) = \Omega p_0 \). Now we will show that \( T_{p_0}H^\nu_r \subseteq V \), i.e., that all the tangent vectors at \( p_0 \) are of the form \( \Omega p_0 \). For that, let \( \gamma(t) \) be an arbitrary curve in \( H^\nu_r \) satisfying \( \gamma(0) = p_0 \). Since \( SO^{+}_{\nu+1}(n+1) \) acts transitively in \( H^\nu_r \), we can say that \( \gamma(t) = R(t)p_0 \), where \( t \rightarrow R(t) \) is a curve in \( SO^{+}_{\nu+1}(n+1) \) satisfying \( R(0) = I \). Therefore, \( \gamma(t) = R(t)p_0 = \Omega(t)R(t)p_0 = \Omega(t)\gamma(0) \), for some curve \( t \rightarrow \Omega(t) \in so_{\nu+1}(n+1) \). Evaluating at \( t = 0 \), we get \( \gamma(0) = \Omega(0)p_0 \) and, consequently, the result holds.

The second equality is obvious and the third trivially follows from the fact that \( \dim((T_{p_0}H^\nu_r)^\perp) = 1 \) and \( p_0 \in (T_{p_0}H^\nu_r)^\perp \).

**Proposition 4.3.** \( \forall p_0 \in H^\nu_r \) and \( 0 \leq \nu \leq n \), we have

\[
T^\nu_{p_0}H^\nu_r \cap H^\nu_r = \{ p_0 + \Omega p_0 : \langle \Omega p_0, \Omega p_0 \rangle = 0 \}.
\]

**Proof.** Let \( p \) be any point in \( T^\nu_{p_0}H^\nu_r \cap H^\nu_r \). Then \( p = p_0 + \Omega p_0 \), with \( \Omega \in so_{\nu+1}(n+1) \), and
\[ \langle p, p \rangle = -r^2. \] Due to the bilinearity and symmetry of \( \langle \cdot, \cdot \rangle \), we can write
\[ \langle p, p \rangle = \langle p_0, p_0 \rangle + 2\langle p_0, \Omega p_0 \rangle + \langle \Omega p_0, \Omega p_0 \rangle. \]
Since \( \langle p_0, p_0 \rangle = -r^2 \) and \( \langle p_0, \Omega p_0 \rangle = 0 \), the condition \( \langle p, p \rangle = -r^2 \) holds only if \( \langle \Omega p_0, \Omega p_0 \rangle = 0. \)

**Remark 4.1.** \( T^\text{aff}_p H^n_v(r) \cap H^n_v(r) = \{ p_0 \} \) if and only if \( \langle \cdot, p_0 \rangle \) restricted to \( T_p H^n_v(r) \) is definite (positive or negative). Therefore, \( \langle \cdot, \cdot \rangle \) is symmetric if and only if \( \nu = 0 \) or \( \nu = n \).

The following figure illustrates the rolling motion of \( H^2_2(r) \) on the tangent plane at \( p_0 \).

The following proposition is analogous for the different situations, so we only present the proof for the curve \( \gamma(t) = p_0 \cos(t) + v \sin(t) \). This case, \( \langle \gamma(t), \gamma(t) \rangle = -r^2 \) which shows that \( \gamma(t) \in H^2_2(r) \). Also, \( \gamma'(t) = -p_0 \cos(t) - v \sin(t) = -\gamma(t) \), so \( \gamma'(t) \in (T_{\gamma(t)} H^2_2(r))^\perp = R^2 \gamma(t) \), \( \forall t \).

**Remark 4.2.** It follows from the last result and from Proposition 4.3 that \( H^2_2(r) \) intersects \( T^\text{aff}_p H^n_v(r) \) along lightlike geodesics containing \( p_0 \).

The following figure shows the three types of geodesics in \( H^2_2(r) \subset R^3 \).

### 5. Kinematic Equations for Rolling Pseudo-Hyperbolic Spaces

Our main result is presented in this section. The kinematic equations for the rolling system we are considering describe the translational and the "rotational" velocity, which are constrained by the no-slip and no-twist conditions. Our strategy is to present a set of differential equations evolving in \( \mathcal{G} = SO_{n+1}^+(n+1) \) and show that their solution is a rolling map.

**Theorem 5.1.** Let \( p_0 \) be an arbitrary point in \( H^2_2(r) \) and \( t \mapsto u(t) \in R^2_0 \) a piecewise smooth function satisfying \( \langle u(t), p_0 \rangle = 0 \). If \( (R(t), s(t)) \in \mathcal{G} \) is the solution of
\[
\begin{align*}
\dot{s}(t) &= r^2 u(t) \\
\dot{R}(t) &= R(t) ( -u(t)p_0^T + p_0 u^T(t) ) J_{n+1},
\end{align*}
\] satisfying the initial condition \( (R(0), s(0)) = (I, 0) \), then \( t \mapsto X(t) = (R^{-1}(t), s(t)) \in \mathcal{G} \) is a rolling map of \( H^2_2(r) \) over its affine tangent space at \( p_0 \), without slipping or twisting, with rolling curve \( t \mapsto \alpha(t) = R(t)p_0 \).

Consequently, equations (6) are the kinematic equations for the rolling motion.

**Proof.** First of all, notice that the second equation in (6) makes sense. Indeed, since the matrix \( -u(t)p_0^T + p_0 u^T(t) \) is skew-symmetric, its multiplication by \( J_{n+1} \) is a matrix belonging to \( s_{0_{n+1}}(n+1) \). Also, the curve \( \alpha(t) = R(t)p_0 \in H^2_2(r) \), \( \forall t \), because \( SO^+(n+1) \) keeps \( H^2_2(r) \) invariant.

Next, we show that all the conditions in the definition of rolling hold for \( X = (R^{-1}, s) \), where \( (R, s) \) is the solution of (refKinematic) satisfying \( (R(0), s(0)) = \).
Checking the Rolling conditions:

The assumption that \( \langle u(t), p_0 \rangle = 0 \), together with the first equation in (6) and the condition \( s(0) = 0 \), guarantees that \( s(t) \in T_{p_0} H^+_0(r) \). So,

\[
\alpha_{dev}(t) = X(t) \circ \alpha(t) = R^{-1}(t) \circ \alpha(t) + s(t) = p_0 + s(t)
\]

and clearly \( \alpha_{dev}(t) \in T_{p_0}^\text{aff} H^+_0(r) \).

The condition that the tangent spaces of the rolling manifolds coincide at each point is easily seen by noticing that both \( T_{\alpha_{dev}(t)} (X(t) \circ H^+_0(r)) \) and \( T_{\alpha_{dev}(t)} (T_{p_0}^\text{aff} H^+_0(r)) \) coincide with \( T_{p_0} H^+_0(r) \).

Checking the no-slip condition

We need to show that

\[
s(t) + R^{-1}(t)(R^{-1}(t)^{-1}(t)(\alpha_{dev}(t) - s(t)) = 0.
\]

Due to (6) and to the expression for \( \alpha_{dev}(t) \), this is equivalent to \( r^2(u(t) + u(t)p_0^T + p_0u^T(t)) = 0 \). Now use the fact that \( \langle p_0, p_0 \rangle = -r^2 \langle u(t), p_0 \rangle = 0 \), to conclude the no-slip condition holds.

Checking the no-twist conditions

For the tangential part, we need to show that

\[
R^{-1}(t)^\circ \dot{R}(t)(\Omega p_0) \in (T_{p_0} H^+_0(r))^\perp, \forall \Omega \in \mathcal{g}_{0,1}(n+1).
\]

Note that \( J_{\nu+1}\Omega \) is a skew-symmetric matrix, which implies that \( p_0^T J_{\nu+1}\Omega p_0 = 0 \). So, also using (6), we can write

\[
R^{-1}(t)^\circ \dot{R}(t)(\Omega p_0) = \langle -u(t)p_0^T + p_0u^T(t) \rangle J_{\nu+1}\Omega p_0
\]

\[
= -u(t)p_0^T J_{\nu+1}\Omega p_0 + p_0u^T(t)J_{\nu+1}\Omega p_0
\]

\[
= \langle u(t), \Omega p_0 \rangle.
\]

But \( \langle u(t), \Omega p_0 \rangle \) is a scalar function, so the condition above holds.

The normal part, reduces to

\[
R^{-1}(t)^\circ \dot{R}(t)(\kappa p_0) \in T_{p_0} H^+_0(r), \forall \kappa \in \mathcal{R}
\]

and it is straightforward to show that it also holds. \( \square \)

Solving the kinematic equations is not an easy task except when the matrix

\[
A(t) := (-u(t)p_0^T + p_0u^T(t)) J_{\nu+1}
\]

is constant. Nevertheless, this matrix function has some interesting properties.

Proposition 5.1. Let \( p_0 \) and \( u(t) \) be as in the statement of Theorem 5.1. Then, for every \( j \in \mathbb{N} \) and every \( t \in \mathcal{R} \), the following holds.

\[
A^{(j-1)}(t) = (r^2u(t)J_{\nu+1}u(t))^j A(t)
\]

and

\[
A^{(j)}(t) = (r^2u(t)J_{\nu+1}u(t))^j A^2(t).
\]

Proof. The proof is straightforward using the assumptions that \( p_0^T J_{\nu+1}p_0 = -r^2 \) and \( u(t)^T J_{\nu+1}u(t) = 0 \). \( \square \)

5.1 Rolling along geodesics

When \( u(t) = u \) is constant, the kinematic equations (6) can be solved explicitly and the corresponding rolling motions are along geodesics. We resume this in the following result.

Theorem 5.2. When \( u(t) = u \) is a (non-zero) constant vector satisfying \( \langle u, p_0 \rangle = 0 \), the solution of the kinematic equations (6), satisfying the initial conditions \( R(0) = I \) and \( s(0) = 0 \), is given by

\[
s(t) = r^2ut, \quad R(t) = e^{tA},
\]

where \( A = (-up_0^T + p_0u^T) J_{\nu+1} \).

Furthermore, the rolling curve \( \alpha(t) = R(t)p_0 \) and its development \( \alpha_{dev}(t) = p_0 + s(t) \) are geodesics in \( H^+_0(r) \) and \( T_{p_0}^\text{aff} H^+_0(r) \), respectively, having the same causal character as the vector \( u \).

Proof. The first part is obvious. For the second, and without loss of generality, normalize \( u \) and consider the three possible situations for its causal character.

- If \( u \) is timelike, say \( \langle u, u \rangle = -1/r^2 \), the Proposition 5.1 can be used to compute

\[
e^{tA} = I + \sin tA + (1 - \cos t)A^2
\]

so that

\[
e^{tA}p_0 = p_0 + \sin tAp_0 + (1 - \cos t)A^2p_0.
\]

But, in this case, \( Ap_0 = r^2u \in A^2p_0 = -p_0 \), hence the rolling curve is

\[
\alpha(t) = p_0 \cos t + ur^2 \sin t.
\]

It is immediate from (3) that \( \alpha(t) \) is a geodesic in \( H^+_0(r) \), while the development curve \( \alpha_{dev}(t) = p_0 + s(t) = p_0 + ur^2t \) is a geodesic in \( T_{p_0}^\text{aff} H^+_0(r) \). These curves satisfy \( \alpha'(0) = \alpha_{dev}'(0) = ur^2 \), hence they are timelike.

- If \( u \) is spacelike, say \( \langle u, u \rangle = 1/r^2 \), a similar procedure enables to conclude that \( \alpha(t) = p_0 \cosh t + ur^2 \sinh t \) and \( \alpha_{dev}(t) = p_0 + ur^2t \), which are spacelike geodesics.

- If \( u \) is lightlike, i.e., \( \langle u, u \rangle = 0 \), then

\[
e^{tA}p_0 = p_0 + tAp_0 + \frac{t^2}{2} A^2p_0.
\]

But in this case \( Ap_0 = r^2u \) and \( A^2p_0 = 0 \), hence the rolling curve and its development coincide and

\[
\alpha(t) = \alpha_{dev}(t) = p_0 + r^2ut.
\]

Clearly \( \alpha \) and \( \alpha_{dev} \) are lightlike geodesics in \( H^+_0(r) \). This may sound strange, but we recall that \( H^+_0(r) \) and its affine tangent space at \( p_0 \) intersect along lightlike geodesics containing \( p_0 \). \( \square \)

6. CONTROLLABILITY OF THE KINEMATIC EQUATIONS

It is clear from Theorem 5.1 that the choice of the function \( u(t) \) determines the rolling curve \( \alpha(t) \) and
vice-versa. So, the kinematic equations (6) can be seen as a control system evolving on the Lie group $SO^+_\nu(n) \ltimes \mathbb{R}^n$. In this section we study the controllability properties of this system, using an approach similar to what has been done for rolling spheres, both in the Euclidean case (Zimmerman, 2005) and in the Lorentzian case (Korolko and Leite, 2011).

Without loss of generality, consider $r = 1$ and $p_0 = (1, 0, \ldots, 0)$, so that $u = (0, u_2, \ldots, u_{n+1})$ and $s = (0, s_2, \ldots, s_{n+1})$. In order to simplify the notation, let $E_{i,j}$ denote the $(n + 1) \times (n + 1)$ matrix with entry $(i,j)$ equal to 1 and the remaining entries equal to zero, $A_{i,j} := E_{i,j} - E_{j,i}$, and $B_{i,j} := E_{i,j} + E_{j,i}$.

For the sake of simplicity, we omit the independent variable $t$ in what follows. The kinematic equations can be written as

$$
\begin{align*}
\dot{s}_2 &= u_2 \\
\vdots \\
\dot{s}_{n+1} &= u_{n+1} \\
\dot{R} &= R\left( \sum_{i=2} \sum_{i=\nu+2} u_i A_{i,1} + \sum_{i=\nu+2} u_i B_{i,1} \right)
\end{align*}
$$

(9)

In order to use available results for controllability on Lie groups, we rewrite this control system in a more convenient form, identifying the states $(R, s)$ with the $(2n + 2) \times (2n + 2)$ matrix

$$
Y = \begin{bmatrix}
R & 0 & s_2 \\
0 & I_n & \vdots \\
0 & \cdots & s_{n+1}
\end{bmatrix}.
$$

In terms of the states $Y$, the control system (9) takes the form

$$
\dot{Y} = Y \left( \sum_{i=2} \sum_{i=\nu+2} u_i A_i \right),
$$

(10)

where

$$
A_i := \begin{cases}
A_{i,1} + E_{n+i,2n+2}, & \text{if } 2 \leq i \leq \nu + 1 \\
B_{i,1} + E_{n+i,2n+2}, & \text{if } \nu + 2 \leq i \leq n + 1.
\end{cases}
$$

We are going to show that for $n > 1$ the system (10) is controllable on the Lie group $SO^+_{\nu+1}(n+1) \ltimes \mathbb{R}^n$.

**Theorem 6.1.** For $n \geq 2$, the control system (10) (or, equivalently, the kinematic equations (6)), which describe the rolling of $H_p^\nu(r)$ over its affine tangent space at $p_0 = (1, 0, \ldots, 0)$, are controllable on $G = SO^+_{\nu+1}(n+1) \ltimes \mathbb{R}^n$.

**Proof.** According to well known results about controllability on Lie groups (see, for instance, Jurdevic and Sussmann, 1972) or, more recently, (Sachkov, 2009)), since $G$ is connected and (10) is a left-invariant control system, without drift term, the system is controllable in $G$ if and only if the Lie algebra generated by $A_{2,\ldots, A_{n+1}}$ is the Lie algebra of $G$, which in this case is $so_{\nu+1}(n+1) \oplus \mathbb{R}^n$.

Therefore, the theorem will be proved if we can obtain all elements in a basis of $L(G)$ as linear combinations of the $A_i$’s and its commutators. For a basis of $L(G)$ we take

$$
\{ A_{i,j} : 1 \leq j < i \leq \nu + 1 \} \cup \{ A_{i,j} : \nu + 2 \leq j < i \leq n + 1 \}
$$

$$
\cup \{ B_{i,j} : 1 \leq j \leq \nu + 1, \nu + 2 \leq i \leq n + 1 \} \cup \{ E_{n+i,2n+2} : 2 \leq i \leq n + 1 \},
$$

(In this set we consider that the first, second and third subsets are empty when $\nu = 0$, $\nu = n - 1$, and $\nu = n$, respectively). The proof involves lots of computations which are difficult to present in a two column format. For that reason, they are not presented here.

\[\square\]

7. CONCLUSION

We derived the kinematic equations for rolling, without slipping or twisting, a pseudohyperbolic space over its affine tangent space at a point. The kinematics are rewritten as a drift-free left-invariant control system evolving on a connected Lie group and controllability of this control system is proved.

8. REFERENCES


