KERNEL BEHAVIORS ON TIME SCALES

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Abstract: This paper is a first attempt to introduce the behavioral approach into dynamical systems define on a time scale. The notion of kernel behavior on a time scale as the kernel of a delta differential matrix operator is introduced. Basic properties, as surjectivity and injectivity, of this operator are studied. A characterization of equivalent kernel representations is given. Additionally, the question whether a behavior that admits a representation with auxiliary variables is also a kernel behavior is analyzed.

Keywords: Dynamical system on time scales, delta differential equations, behavioral systems, kernel representations, equivalence of representations

1. INTRODUCTION

The behavioral approach to systems and control theory was introduced by J.C.Willems (Willems, 1986; Willems, 1989; Willems, 1991) The main idea of this theory is to foucus on the set of trajectories of a system, rather then the equations that may be taken in order to describe the phenomena. Furthermore, the variables of the system are not, *a priori*, divided into inputs and outputs. This approach leads to general and comprehensive definition of a dynamical system as a triple $(T, W, \mathcal{B})$ where T is a set, called the time set, W an other set (the signal set) and $\mathcal{B}$ a subset of functions from T to W. $\mathcal{B}$ is called the behavior of the system, and each element of $\mathcal{B}$ is said to be a trajectory of the system (behavior). Typically, the time set T is $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ or $\mathbb{Z}_+$. In behavioral systems theory, one of the central and basic questions is to characterize the behaviors that are solution sets of systems of linear difference/differential equations with constant coefficients, in particular homogeneous ones. These behaviors are usually called kernel behaviors.

The aim of this paper is to give some preliminary analysis on kernel behaviors from the perspective of the theory of time scales. Calculus on time scales, developed by (Hilger, 1988), unifies differential calculus on the real line and calculus of finite differences. A time scale is a closed subset of $\mathbb{R}$, in which are defined two basic operators: forward jump and a backward jump, and differential operators that generalize the usual difference and differential operator. Usually, the differential operator in consideration is the delta differential operator.

Dynamical systems on time scales were studied in (Bohner and Peterson, 2001) and (Bohner and Peterson, 2003). The basic aspects of the theory of linear control systems on a time scale were developed as well in, for example, (Bartosiewicz and Pawluszewicz, 2004; Bartosiewicz and Pawluszewicz, 2006; Davis *et al.*, 2009).

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In the paper we introduce systems $(\mathbb{T}, W, \mathcal{B})$ where $\mathbb{T}$ is a time scale, with $\sup \mathbb{T} = \infty$, and $\mathcal{B}$ is the solution set of a system of homogeneous linear delta differential equations. These systems may be regarded as kernels of delta differential matrix operators and for that reason they are called kernel behaviors on the time scale $\mathbb{T}$. As in traditional behavior theory, to each delta differential matrix operator it is associated a polynomial matrix. The polynomial matrix (delta differential matrix operator) is called a kernel representation of the system. We study some basic properties of this kind of systems. Namely, we deal with kernel representation issues. We also analyze the question whether a behavior on a time scale with a representation with auxiliary variables is also a kernel behavior. The obtained results, under some additional condition on the representations (regressivity), extend existing results for continuous time case, see (Polderman and Willems, 1997; Vincente, 1998). Similar properties are obtained by imposing additional conditions on the time scale. Notice that, in time scales approach, the delta differential operator is used in order to extend and unify the usual differential operator (in $\mathbb{R}$) and the forward difference operator (in $\mathbb{Z}$). So, in an arbitrary time scale the behavior of this operator may differ from the usual operators.

The paper is organized as follows. In Section 2 basic ideas of time scale theory are given. In Section 3 the problem of existence of general solutions for scalar delta differential equations is studied. The foundations of system behaviors on time scales are given in Section 4. In Section 5 is discussed the problem of representation equivalence. In Section 6 the representations with auxiliary variables of a dynamical system on a time scale is studied. Finally, in Section 7 some final remarks are done.

2. PRELIMINARIES ABOUT TIME SCALES

In this section we recall some basic facts about time scales. More about time scale calculus can be found in (Bohner and Peterson, 2001).

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. If $a, b \in \mathbb{T}$ and $a < b$, we distinguish $[a, b]$ as a real interval and we define $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. On $\mathbb{T}$ are taken the usual jump operators, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ such that $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, if $t \neq \max \mathbb{T}$, $\sigma(\max \mathbb{T}) = \max \mathbb{T}$, and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ such that $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, if $t \neq \min \mathbb{T}$, $\rho(\min \mathbb{T}) = \min \mathbb{T}$. Using these operators, points in real line can be classified as follows: if $\sigma(t) > t$, then t is called *right-scattered*, if $\rho(t) < t$ is called *left-scattered*, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*, if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*. From the jump operators, the *graininess functions* $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, respectively. These function together with the jump operators play an important role, namely on the time scales classification and analysis.

Example 1.

(1) $\mathbb{R}$ is a time scale where $\rho(t) = t = \sigma(t)$ and $\nu(t) = \mu(t) = 0$, for all $t \in \mathbb{R}$.
(2) Another time scale with constant graininess is $c\mathbb{Z}$, with $c > 0$, where $\rho(t) = t - c$, $\sigma(t) = t + c$ and $\nu(t) = \mu(t) \equiv c$, for all $t \in c\mathbb{Z}$.
(3) $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, with $q > 1$, is also a time scale. In this case, $\rho(t) = \frac{t}{q}$, $\sigma(t) = qt$ and $\nu(t) = (1 - \frac{1}{q})t$, $\mu(t) = (q-1)t$, for all $t \in \mathbb{T}$.

A time scale with constant graininess function μ is called a *homogeneous* time scale. Time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = c\mathbb{Z}$, $c > 0$, $[a, +\infty[$, $a \in \mathbb{R}$, are homogeneous while, for example $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ is not.

If $\mathbb{T}$ has a left–scattered $\max \mathbb{T} = b$, then we define $\mathbb{T}^\kappa = \mathbb{T} - \{b\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2. ((Bohner and Peterson, 2001)). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The *delta derivative* of f at t, denoted by $f^\Delta(t)$ (or by $\frac{\Delta}{\Delta t} f$), is the real number (provided it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

If $x \in \mathbb{T} \setminus \mathbb{T}^\kappa$, then f^Δ is not uniquely defined, since for such x, small neighborhoods of x consist only of x and besides we have $\sigma(x) = x$. Therefore inequality in Definition 2 holds for an arbitrary number $f^\Delta(x)$. This is a reason why we omit a maximal left-scattered point.

Example 3.

(1) If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if f is differentiable in the ordinary sense at t. In this case, $f^\Delta(t) = \lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$.
(2) If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable on $\mathbb{Z}$. Furthermore, $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$.

For a function $f : \mathbb{T} \to \mathbb{R}$ the second delta derivative is defined as $f^{[2]} = \frac{\Delta^2}{\Delta^2 t} f := (f^\Delta)^\Delta$, provided that f^Δ is delta differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$ with delta derivative $f^{[2]} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Furthermore, $f^{[k]}$ denotes the delta derivative of f of order k, whenever it exists, and $f^{[0]} := f$.

In this paper we use the Hilger generalized exponential function, $e_p(t,t_0)$, defined for *regressive* functions $p\colon \mathbb{T} \to \mathbb{C}$, (Hilger, 1990). A function $p\colon \mathbb{T} \to \mathbb{C}$ is said to be *regressive* if $1+\mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}$. The Hilger exponential is defined for a complex valued regressive function p as follows:

$$e_p(t,t_0) = \exp\left[\int_{t_0}^{t} \xi_{\mu(t)}(p(\tau))\Delta\tau\right],$$

where $\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}\,(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0 \end{cases}$.

3. SCALAR LINEAR EQUATIONS ON TIME SCALES

We are going to deal with liner delta differential equations defined on a time scale $\mathbb{T}$ such that $\mathbb{T}^\kappa = \mathbb{T}$ (that is $\sup \mathbb{T} = +\infty$), i.e, sets of solutions of equations of the following type

$$a_n y^{[n]}(t) + \cdots + a_1 y^{\Delta}(t) + a_0 y(t) = b(t) \qquad (1)$$

where $a_i \in \mathbb{R}$, $b\colon \mathbb{T} \to \mathbb{R}$, for $i = 1,\ldots,n$, $y\colon \mathbb{T} \to \mathbb{R}$ and $a_n \neq 0$. The function y is said to be a solution of (1) if y satisfies (1). Here, we consider functions y that have infinitely many derivatives in $\mathbb{T}$. It is well known that the Hilger generalized exponential function, $e_p(t,t_0)$, $p : \mathbb{T}\to\mathbb{C}$, is a solution of the equation

$$y^{\Delta}(t) - py(t) = 0\,, \qquad (2)$$

if p is regressive, (Bohner and Peterson, 2001, p. 61), and it is the unique one that satisfies $y(t_0) = 1$.

Definition 4. The equation (1) is regressive if $\sum_{k=0}^{n} a_k(-\mu(t))^{n-k} \neq 0$, for all $t \in \mathbb{T}$.

Note that, equation (2) is regressive if and only if p is regressive. The notion of regressivity provides a sufficient condition for the existence of solution of (1), for a fixed *rd-continuous function* $b(t)$, as stated in the next proposition, (Bohner and Peterson, 2001, pp. 190, 239). A function $b\colon \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (in $\mathbb{R}$) at left-dense points in $\mathbb{T}$.

Proposition 1. If (1) is regressive, then there exists a solution of (1), for each rd-continuous function $b(t)$. Furthermore, the solution is unique for initial conditions $y^{[k-1]}(t_0) = y_{k-1}$, $k = 1,2,\ldots,n$.

The regressivity of (1) may also be characterized as follows, (Bohner and Akin-Bohner, 2003).

Proposition 2. Equation (1) is regressive if and only if each root of the polynomial $r(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$ is a regressive function, *i.e*, $1+\lambda\mu(t) \neq 0$, for all $t \in \mathbb{T}$, for each $\lambda \in \mathbb{C}$ such that $r(\lambda) = 0$.

Next we introduce the definition of regressive polynomial with respect to a fixed time scale.

Definition 5. A polynomial $r(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is said to be regressive with respect to a time scale $\mathbb{T}$ if its roots are regressive.

We may drop the regressivity condition in Proposition 1 in a time scale $\mathbb{T}$ where $\inf \mathbb{T} = t_0$ is finite. This is a consequence of a local existence result, see (Bohner and Peterson, 2001, pp.322), that enables to construct uniquely defined "forward" solutions. In particular, for these type of time scale, there exists and it is unique the solution of equation (2) for the initial condition $y(t_0) = 1$. In this case, even if the equation is not a regressive one, such solution is also denoted by $e_p(t,t_0)$.

Proposition 3. If $\mathbb{T}$ is such that $\min \mathbb{T} = t_0$, then there exists a solution of (1), for each rd-continuous function $b(t)$. Furthermore, the solution is unique for initial conditions $y^{[k-1]}(t_0) = y_{k-1}$, $k = 1,2,\ldots,n$.

For homogeneous time scales it is also possible to guarantee the existence of solution for equation (1) without the regressivity condition. However, in a homogeneous time scale (if we drop the regressivity condition) not every initial value problem has a solution.

Example 6. In $\mathbb{Z}$ the initial value problem $y^{\Delta} + y = 0$, $y(0) = 1$, does not have solution, since the equation $y^{\Delta} + y = 0$ only admits the solution $y(t) \equiv 0$.

In the proof of the following proposition we show how to deal with the non-regressivity of (1) in case of homogeneous time scales.

Proposition 4. If $\mathbb{T}$ is homogeneous, then there exists a solution of (1), for each rd-continuous function $b : \mathbb{T}\to\mathbb{R}$.

PROOF. Let $r(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$ be the polynomial associate to (1) and $\mu(t) = h$ the graininess of $\mathbb{T}$. If r is regressive the statement is true, as we have seen. If r is non regressive, $h \neq 0$ and $-\frac{1}{h}$ is a root of r, with a certain multiplicity k. That is, $p(s) = (s+\frac{1}{h})^k \tilde{p}(s)$, where $\tilde{p}(s)$ is a regressive polynomial of degree $n-k$. So, the Δ-differential equation (1) takes the following form

$$\left(\Delta + \frac{1}{h}\right)^k \tilde{p}(\Delta)y(t) = b(t) \qquad (3)$$

If one puts

$$\tilde{p}(\Delta)y(t) = \tilde{y}(t), \qquad (4)$$

then equation (3) can be written as $\left(\Delta + \frac{1}{h}\right)^k \tilde{y}(t) = b(t)$ which is equivalent to $\frac{1}{h^n}\tilde{y}(t+nh) = b(t)$, that is

$$\tilde{y}(t + nh) = h^n b(t) \quad (5)$$

for every $t \in \mathbb{T}$. Two cases must be addressed:

1) $t - nh \in \mathbb{T}$ for every $t \in \mathbb{T}$,
2) $t - nh \notin \mathbb{T}$ for certain $t \in \mathbb{T}$.

In the case 2) the time scale $\mathbb{T}$ has a minimum point t_0, so by Proposition 3, equation $p(\Delta)y(t) = b(t)$ has a solution for every rd-continuous function $b(t)$.

In case 1), equation (5) is equivalent to $\tilde{y}(t) = h^n b(t-nh)$. For this $\tilde{y}(t)$, by Proposition 1, equation (4) as a solution. That solution is also a solution of (1). □

4. REPRESENTATIONS OF DYNAMICAL SYSTEMS ON TIME SCALES

In the following, $\mathbb{T}$ represents a time scale such that $\mathbb{T}^\kappa = \mathbb{T}$. Assume that W is a set of infinitely many time delta differentiable functions $w : \mathbb{T} \to \mathbb{R}^q$. The triple $\Sigma = (\mathbb{T}, \mathbb{R}^q, \mathcal{B})$, where $\mathcal{B} \subseteq W^{\mathbb{T}}$, is called a *dynamical system*. As usual, the set $\mathcal{B}$, which is the family of functions of time from $\mathbb{T}$ to $\mathbb{R}^q$ that formalizes the laws of the system, is called the *system behavior*. Typically, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, (Willems, 1989; Willems, 1991).

The system Σ is said to be linear, if the behavior $\mathcal{B}$ is a vector space (over $\mathbb{R}$) and $\mathcal{B}$ is a linear subspace of $W^{\mathbb{T}} = \{w : \mathbb{T} \to \mathbb{R}\}$. Thus, linear systems obey the superposition principle, *i.e.*, $\{w_1(\cdot), w_2(\cdot) \in \mathcal{B}, \alpha, \beta \in \mathbb{R}\} \Rightarrow \{\alpha w_1(\cdot) + \beta w_2(\cdot) \in \mathcal{B}\}$.

Let us consider dynamical systems $\Sigma = (\mathbb{T}, \mathbb{R}^q, \mathcal{B})$, where $\mathbb{T}$ is an infinite time scale such that $\mathbb{T}^\kappa = \mathbb{T}$ and $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{T}}$ is the set of solutions of a linear Δ-differential equation

$$R_L w^{[L]}(t) + R_{L-1} w^{[L-1]}(t) + \cdots R_0 w(t) = 0 \quad (6)$$

where $R_L, R_{L-1}, \ldots, R_0 \in \mathbb{R}^{g \times q}$ and $R_L \neq 0$. So in consideration are behaviors of the following type

$$\mathcal{B} = \{w\colon \mathbb{T} \to \mathbb{R}^q : R_L w^{[L]}(t) + R_{L-1} w^{[L-1]}(t) + \cdots + R_0 w(t) = 0, \text{ for all } t \in \mathbb{T}\}. \quad (7)$$

A behavior $\mathcal{B}$ that can be written as in (7) is called a *kernel behavior on time scale* $\mathbb{T}$.

Proposition 5. Let $\Sigma = (\mathbb{T}, \mathbb{R}^q, \mathcal{B})$ be a dynamical system. If $\mathcal{B}$ is a kernel behavior (defined as in (7)), then Σ is linear.

PROOF. Linearity comes straightforward from the fact that Δ is a linear operator, in the sense that $(f+g)^\Delta = f^\Delta + g^\Delta$ and $(\alpha f)^\Delta = \alpha f^\Delta$. □

Equation (6) can be rewritten in the form

$$R(\Delta)w = 0 \quad (8)$$

where Δ denotes the delta-derivative and $R \in \mathbb{R}^{g \times q}[s]$ is the polynomial matrix defined by $R(s) := R_L s^L + R_{L-1} s^{L-1} + \ldots + R_0$. As usual, $R(s)$ and (8) are called a matrix representation of $\mathcal{B}$ and a kernel representation of $\mathcal{B}$, respectively. In this case, $\mathcal{B}$ is also denoted by $\mathcal{B}(R)$. A kernel behavior has infinitely many matrix representations. In particular, if $U(s)$ is an unimodular matrix $\mathcal{B}(UR) = \mathcal{B}(R)$. This is due to the fact that the set of solutions of system of linear Δ-differential equations is preserved by the following elementary operations over equations:

I. Interchanging of equations;
II. Multiplying an equation by a nonzero real number;
III. Adding to an equations an other Δ-differentiated.

Two polynomial matrices $R \in \mathbb{R}^{g_1 \times q}[s]$ and $T \in \mathbb{R}^{g_2 \times q}[s]$ are said to be equivalent if $\mathcal{B}(R) = \mathcal{B}(T)$. In particular two polynomial matrices $R(s)$ and $T(s)$ with the same numbers of rows are said to be *unimodularly equivalent representations* if there exists an unimodular matrix $U(s)$ such that $R(s) = U(s)T(s)$. As in the traditional behavior theory over $\mathbb{R}$ (or $\mathbb{Z}$), a system over an arbitrary time scale has always a representation with full row rank. These representations are usually called regular. In fact, if a matrix $R(s)$ does not have full row rank than there exists an unimodular matrix $U(s)$ such that $U(s)R(s) = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$, where $T(s)$ is full row rank. It is easy to see that $\mathcal{B}(R) = \mathcal{B}(T)$.

5. EQUIVALENT KERNEL REPRESENTATIONS

In the classical cases of $\mathbb{R}$ or $\mathbb{Z}$, it is a well known fact that if $R_1(s)$ and $R_2(s)$ are polynomial matrices with full row rank, $\mathcal{B}(R_1) = \mathcal{B}(R_2)$ if and only if R_1 and R_2 are unimodularly equivalent. For arbitrary time scale, one of the implications does not hold without further restrictions. In fact, from the previous remarks it is straightforward that if R_1 and R_2 are unimodularly equivalent then $\mathcal{B}(R_1) = \mathcal{B}(R_2)$. But, in some time scales, it can happen that two regular representations of the same behavior may not be unimodularly equivalent. A simple example is the following.

Example 7. In $\mathbb{Z}$ let us that the kernel behavior defined by $w^\Delta + w = 0$. Notice that, this kernel is the null space. So $R(s) = s + 1$ and $I(s) = 1$ are regular representations of the same behavior but, they are not unimodularly equivalent.

Given a polynomial matrix $R(s) \in \mathbb{R}^{g \times q}[s]$ there exist unimodular polynomial matrices $U(s)$ and $V(s)$ such that $U(s)R(s)V(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}$, where $D(s) = \operatorname{diag}(d_1(s), \ldots, d_r(s))$ with $r \leq \min(q, g)$, for some monic polynomials $d_i(s)$, $i = 1, \ldots, r$, such that d_i divide d_{i+1}, for $i = 1, \ldots, r-1$. These polynomials are called *invariant polynomials* of $R(s)$ and

$\begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}$ is called the *Smith canonical form of* $R(s)$. Also, notice that r is the rank of $R(s)$ as a polynomial matrix. This rank is denoted by $\mathrm{rank}_{\mathbb{R}[s]} R(s)$.

Definition 8. Let $R(s) \in \mathbb{R}^{g\times q}[s]$ be a polynomial matrix with rank r and let $\mathbb{T}$ be a time scale. R is said to be regressive with respect to $\mathbb{T}$ if its invariant polynomials $d_1(s), \ldots, d_r(s)$ are regressive polynomials with respect to $\mathbb{T}$.

Each of the following propositions is an extension version of the classical ones for the continuous and discrete cases. Their proofs run similarly to the ones of the classical case, see (Willems, 1989; Vincente, 1998). However, special care must be used in the steps where Δ-differential operator and time scale properties are in question. The proofs that we present are strongly inspired by the ones made for $\mathbb{T} = \mathbb{R}$ in (Vincente, 1998). This option was made in order to clarify the necessary modifications and to improve the readability of the paper, specially for those not familiar with the behavioral approach. However, due to space constrains, some parts of the proofs will be skipped.

Proposition 6. Let $R(\Delta) : C^\infty(\mathbb{T}, \mathbb{R}^q) \to C^\infty(\mathbb{T}, \mathbb{R}^g)$ be Δ-differential operator such that $R(s)$ is regressive. Then the following hold:

i) $R(\Delta)$ is surjective if and only if $\mathrm{rank}_{\mathbb{R}[s]} R(s) = g$,
ii) $R(\Delta)$ is injective if and only if $\mathrm{rank}_{\mathbb{C}} R(\lambda) = q$ for all $\lambda \in \mathbb{C}$,
iii) $R(\Delta)$ is bijective if and only if $R(s)$ is unimodular.

PROOF. Part $i)$ *"only if"*: Since a kernel behavior, in every time scale, has always a regular representation, this proof runs as in $\mathbb{T} = \mathbb{R}$. Therefore, the regressivity condition is not needed here.
Part $i)$ *"if"* : Using de Smith form of $R(s)$, there exist unimodular matrices $U(s)$ and $V(s)$ such that $U(s)R(s)V(s) = \begin{bmatrix} D(s) & 0_{g\times(q-g)} \end{bmatrix}$, where $D(s) = \mathrm{diag}(d_1(s), \ldots, d_g(s))$ and $d_i(s) \neq 0$ for $i = 1, \ldots, g$. Then the equation

$$R(\Delta)w = a \tag{9}$$

is equivalent to $\begin{bmatrix} D(\Delta) & 0_{g\times(q-g)} \end{bmatrix} v = u$, where $v = V^{-1}(\Delta)w$ and $u = U(\Delta)a$. If $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ where $v_1 := \begin{bmatrix} v_{11} & \ldots & v_{1g} \end{bmatrix}^T$ and $u := \begin{bmatrix} u_1 & \ldots & u_g \end{bmatrix}^T$, the equation (9) can be rewritten as

$$\begin{aligned} d_1(\Delta)v_{11} &= u_1 \\ &\vdots \\ d_g(\Delta)v_{1g} &= u_g. \end{aligned} \tag{10}$$

Since, by Proposition 3, equations in (10) have solutions for every $u_i \in C^\infty(\mathbb{T}, \mathbb{R})$, then also equation (9) has a solution for every a.

Part $ii)$ *"only if"*:
Let us assume that $\mathrm{rank}_{\mathbb{C}} R(\lambda) < q$ for some $\lambda \in \mathbb{C}$. Then, for such a $\lambda \in \mathbb{C}$, there exists a nonzero vector $\alpha \in \mathbb{R}^q$ such that $R(\lambda)\alpha = 0$. Since $R(s)$ is regressive, then λ is also regressive. Considering a nonzero function $w(t) = \alpha e_\lambda(t, t_0)$, with $t \in \mathbb{T}$ and $t_0 \in \mathbb{T}$ arbitrary chosen, and taking into account that $(e_\lambda(t, t_0))^\Delta = \lambda e_\lambda(t, t_0)$, we can see that $R(\Delta)\alpha e_\lambda(t, t_0) = 0$. This implies that $R(\Delta)$ is not injective.

Part $ii)$ *"if"*:
This proof is also made using Smith form and runs as in $\mathbb{T} = \mathbb{R}$, for every time scale (regressivity condition is not need).

Part $iii)$ is a direct consequence of Parts $i)$ and $ii)$. □

Note that Proposition 6 can be also be stated without regressivity condition, if we restrict ourselves to some type of time scales where every equation of the form (2) is uniquely solvable for a nonzero initial condition in a certain point $t_0 \in \mathbb{T}$. Namely, in view of Proposition 4, for time scales that have minimum point.

Lemma 9. Let $R_1(s) \in \mathbb{R}^{g_1\times q}[s]$ and $R_2(s) \in \mathbb{R}^{g_2\times q}[s]$ be regressive matrices. If $\mathcal{B}(R_1) \subseteq \mathcal{B}(R_2)$ then there exist a polynomial matrix $M(s)$ such that $R_2(s) = M(s)R_1(s)$.

PROOF. The proof may be done under the assumption that R_1 is full row rank, since if it is not, we may take a regular one. Moreover, under regressivity assumption, this proof runs as in $\mathbb{T} = \mathbb{R}$. In fact, it can be obtained from Proposition 6, using some properties of polynomial matrices. □

The following proposition can be obtained from Lemma 9 using classical techniques only involving polynomial matrices. For this reason we omit its proof.

Proposition 7. Let $R_1(s) \in \mathbb{R}^{n_1\times g}[s]$ and $R_2(s) \in \mathbb{R}^{n_2\times g}[s]$ be polynomial matrices with full row rank. If R_1 and R_2 are regressive with respect to $\mathbb{T}$, then $\mathcal{B}(R_1) = \mathcal{B}(R_2)$ if and only if R_1 and R_2 are unimodularly equivalent.

6. REPRESENTATION WITH AUXILIARY VARIABLE

In many modeling problems it is convenient to introduce an auxiliary variable, here denoted by ξ. As w, we consider that ξ has infinitely many delta-derivatives. This leads to the equation

$$R'(\Delta)w = R''(\Delta)\xi \tag{11}$$

where $R' \in \mathbb{R}^{g\times q}[s]$, $R'' \in \mathbb{R}^{g\times f}[s]$. The behavior of the system described by equation (11) is

$$\mathcal{B}_{AUX}(R', R'') = \{w : \mathbb{T}\to\mathbb{R}^q : \ \exists\xi : \mathbb{T}\to\mathbb{R}^f \\ \text{such that } R'(\Delta)w = R''(\Delta)\xi\}.$$

So, it is natural to ask if a behavior with an auxiliary variable representation of type (11) has also a kernel representation. This is known to be true for the traditional times scales as $\mathbb{R}$ or $\mathbb{Z}$.

Notice that, from a carefully made analysis of the proof of Proposition 6, part (i), we may derive the following corollary.

Corollary 8. Let $P \in \mathbb{R}^{n\times p}[s]$ be a full row rank polynomial matrix. Then

i) if $\mathbb{T}$ is a homogeneous time scale, then for every $a \in C^\infty(\mathbb{T}, \mathbb{R}^n)$ there exists $\xi \in C^\infty(\mathbb{T}, \mathbb{R}^p)$ such that $P(\Delta)\xi = a$,
ii) if $\min \mathbb{T} = t_0$ then for every $a \in C^\infty(\mathbb{T}, \mathbb{R}^n)$ there exists $\xi \in C^\infty(\mathbb{T}, \mathbb{R}^p)$ such that $P(\Delta)\xi(t) = a$ for all $t \in [t_0; \sup \mathbb{T})_{\mathbb{T}}$.

Applying the previous corollary and similar arguments to the ones used for $\mathbb{T} = \mathbb{R}$ it is possible to prove the following (the proof is omitted).

Proposition 9. Let $(\mathbb{T}, \mathbb{R}^q, \mathcal{B})$ be a dynamical system such that $\mathcal{B} \subset C^\infty(\mathbb{T}, \mathbb{R}^q)$. If $\mathbb{T}$ is homogeneous or has a minimum point, then $\mathcal{B} = \mathcal{B}(R)$, for some polynomial matrix R, if and only if $\mathcal{B} = \mathcal{B}_{AUX}(R', R'')$, for some polynomial matrices R' and R''.

Using a similar proof and Proposition 6, part (i), we also may state the next proposition.

Proposition 10. Let $(\mathbb{T}, \mathbb{R}^q, \mathcal{B})$ be a dynamical system such that $\mathcal{B} \subset C^\infty(\mathbb{T}, \mathbb{R}^q)$. $\mathcal{B} = \mathcal{B}_{AUX}(R', R'')$, for some polynomial matrices $R'(s)$ and $R''(s)$, with R'' regressive, if and only if $\mathcal{B} = \mathcal{B}(R)$, for some polynomial matrix R.

7. CONCLUSION

In this paper we have proposed a study of dynamic systems on time scales basics using the behavioral approach. We have introduced the notion of kernel behavior on a time scale and analyzed questions related to its representations.

In order to lay out some kind of foundations to the theory of kernel behaviors on a time scale, we have made a deep analysis of both discrete and continuous cases in the classical behavioral theory. As usual, in time scales, our aim was to propose some extension and unification of those cases. We have shown that the regressivity condition on the representation of the behavior provides in most cases a framework that allows to work in an arbitrary time scale as in the continuous case. We have also considered the case where the regressivity condition is not assumed but additional constrains are imposed to the time scale. Namely, we have considered homogeneous time scales and time scales with minimum point. These restrictions (regressivity, homogeneity and minimum point existence) are related to the surjectivity and injectivity of the polynomial Δ-differential operator. This can be specially notice in Section 3, in the proof of Proposition 6 and subsequent comments.

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